Overview

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    - periodogram-based frequency estimation
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spectrum estimation

- *introduction*
  - objective is to estimate the power spectral density of a wide-sense stationary random process,
    - the power spectrum is the Fourier transform of the autocorrelation sequence
  - for an ergodic process, the autocorrelation may be estimated as

$$\boxed{r_X[k] = \lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{\ell=-N}^{N} x[\ell]\,x^*[\ell-k]\right\} \quad \xleftrightarrow{F} \quad P_X\left(e^{j\omega}\right)}$$

two difficulties exist:

- the amount of data is finite
- data is often corrupted by noise or contaminated with an interfering signal

∴ spectrum estimation is a problem that involves estimating $P_X(e^{j\omega})$ from a set of noisy measurements of $x[n]$

- spectrum estimation is important in many areas including
  - signal detection and tracking (e.g. sonar)
  - harmonic analysis and prediction
  - time series extrapolation, spectral smoothing, bandwidth compression
  - beamforming and direction finding

spectrum estimation

- *introduction*
  - two main classes of approaches to spectrum estimation:
    - non-parametric methods, which include the periodogram, the modified periodogram, Bartlett's method, Welch's method, and the Black-Tukey method
    - parametric methods, which include a specific model to the data such as AR, MA, and ARMA
  - we will also consider the minimum variance method which involves power spectrum estimation at the center of each band-pass filter of a narrow-band filter bank, as well as the maximum entropy method (MEM) that presumes the all-pole model
  - the problem of frequency estimation for harmonic processes that consist of a sum of sinusoids or complex exponentials in noise, is also addressed and involve such algorithms as Pisarenko harmonic decomposition, MUSIC, the eigenvector method and the minimum norm algorithm
  - a quick perspective is also given on principal components frequency estimation

spectrum estimation

- *non-parametric methods: the periodogram*
  - the periodogram is easy to compute but is limited in its ability to produce an accurate estimate of the power spectrum, especially for short data records

by definition $P_X\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty} r_X[k]\,e^{-jk\omega}$ which denotes that the power spectrum estimation problem is an autocorrelation estimation problem; when x[n] is measured over a limited time internal, n=0,1,…,N-1, then

$$\hat{r}_X[k]=\frac{1}{N}\sum_{\ell=0}^{N-1} x[\ell]\,x^*[\ell-k]$$

excluding all data values outside the n=0,1,…,N-1 interval,

$$\hat{r}_X[k]=\frac{1}{N}\sum_{\ell=k}^{N-1} x[\ell]\,x^*[\ell-k], \qquad k=0,1,\ldots,N-1$$

Note: this is a biased estimator, an unbiased estimator would divide by N-k, instead of just N

and $$\hat{r}_X[-k]=\hat{r}_X^*[k], \qquad |\hat{r}_X[k]|=0, \qquad k\geq N$$

spectrum estimation

the periodogram is obtained as $\hat{P}_{PER}\left(e^{j\omega}\right)=\sum_{k=1-N}^{N-1}\hat{r}_W[k]e^{-jk\omega}$

it may also be conveniently expressed as a function of x[n] by considering a windowed representation of the signal

$$x_W[n]=w[n]\,x[n],\qquad w[n]=0,\, n<0 \vee n>N-1$$

therefore $\hat{r}_X[k]=\frac{1}{N}\sum_{\ell=-\infty}^{\infty}x_W[\ell]\,x_W^*[\ell-k]=\frac{1}{N}x_W[k]*x_W^*[-k]$

which means, according to the convolution theorem

$$\hat{P}_{PER}\left(e^{j\omega}\right)=\frac{1}{N}X_W\left(e^{j\omega}\right)X_W^*\left(e^{j\omega}\right)=\frac{1}{N}\left|X_W\left(e^{j\omega}\right)\right|^2$$

where $X_W\left(e^{j\omega}\right)=\sum_{k=-\infty}^{\infty}x_W[k]e^{-jk\omega}\underset{\substack{w[k]=1,\\ k=0,1,\ldots,N-1}}{=}\sum_{k=0}^{N-1}x[k]e^{-jk\omega}$

thus, the periodogram is proportional to the squared magnitude of the DTFT of $x_W[n]$ and may be easily computed using a DFT:

$$x[n] \rightarrow x_W[n] = w[n]x[n] \xrightarrow{DFT} X_W[k] \rightarrow \frac{1}{N}\left|X_W[k]\right|^2 = \hat{P}_{PER}\left(e^{jk2\pi/N}\right), \quad k = 0,1,\ldots N-1$$

The periodogram has an interesting interpretation in terms of filter banks

As discussed in the last class, the DFT may be regarded as a bank of filters whose impulse responses are generally given by

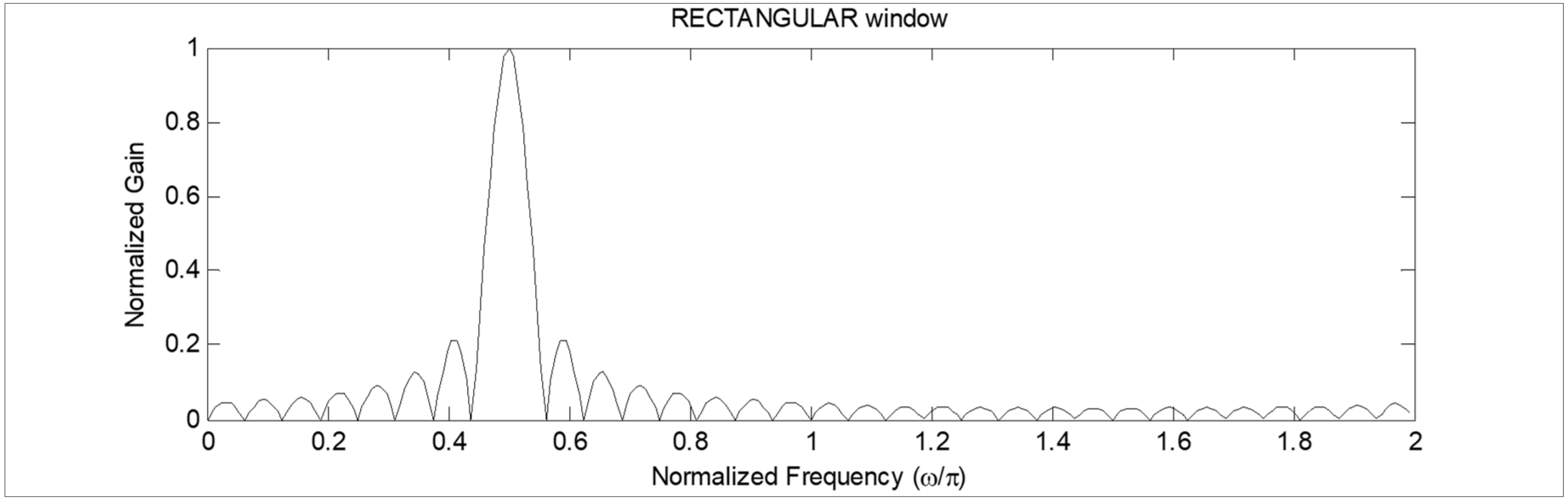
$$h_K[n] = e^{-j\omega_k(N-1)} w[n] e^{j\omega_k n}, \qquad \omega_k = k\frac{2\pi}{N}, \quad n,k = 0,1,\ldots N-1$$

and the corresponding frequency responses, when w[n] is the rectangular window, are given by

$$H_K\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h_K[n] e^{-j\omega n} = e^{-j(\omega+\omega_k)\frac{N-1}{2}} \frac{\sin\left(\omega-\omega_K\right)\frac{N}{2}}{\sin\left(\omega-\omega_K\right)\frac{1}{2}}$$

spectrum estimation

an example of the normalized magnitude of this frequency response is represented next for channel k=8, when N=32



Therefore, each channel of the DFT may be regarded as the output of a band-pass filter which is centered at $\boxed{\omega_K = k2\pi / N}$ and whose bandwidth is approximately $\boxed{\Delta\omega \approx 2\pi / N}$ in case w[n] is the rectangular window

As also derived in the last class, the output of the k-th channel (or sub-band) of the DFT filter bank may be characterized as

$$\boxed{y_K[n] = x[n] * h_K[n] = \sum_{\ell=n-(N-1)}^{n} x[\ell] h_K[n-\ell]}$$

spectrum estimation

by normalizing the gain of the output of each sub-band (simply achieved by dividing by N), then, the power spectrum of x[n] and y[n] are equal at frequency ω_K, i.e.

$$\hat{P}_X\left(e^{j\omega_K}\right) = \hat{P}_Y\left(e^{j\omega_K}\right)$$

if the bandwidth is small enought so that the power spectrum of x[n] is approximately constant over the pass band of the filter, then the power in $y_K[n]$ is approximately

$$E\left\{\left|y_K[n]\right|^2\right\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_X\left(e^{j\omega}\right)\left|H_K\left(e^{j\omega}\right)\right|^2 d\omega \approx \frac{\Delta\omega}{2\pi} P_X\left(e^{j\omega_K}\right) = \frac{1}{N} P_X\left(e^{j\omega_K}\right)$$

and, therefore, by estimating the power in $y_K[n]$ as $\hat{E}\left\{\left|y_K[n]\right|^2\right\} = \left|y_K[N-1]\right|^2$

$$\hat{P}_X\left(e^{j\omega_K}\right) = N\,\hat{E}\left\{\left|y_K[n]\right|^2\right\} = \frac{1}{N}\left|\sum_{\ell=0}^{N-1} x[\ell]e^{-j\ell\omega_K}\right|^2 = \frac{1}{N}\left|X_W[k]\right|^2$$

which is equivalent to the periodogram.

spectrum estimation

- *performance of the periodogram*

in order to be a *consistent* estimator of the power spectrum, the mean-square convergence of the periodogram must be asymptotically unbiased, i.e.

$$\lim_{N\to\infty} E\left\{\left[\hat{P}_{PER}\left(e^{j\omega}\right)-P_X\left(e^{j\omega}\right)\right]^2\right\}=0$$

and the variance should go to zero as the data lenght N tends to infinity, i.e.

$$\lim_{N\to\infty} Var\left\{\hat{P}_{PER}\left(e^{j\omega}\right)\right\}=0$$

Concerning *bias*, we evaluate first the expected value of the estimated autocorrelation

$$E\left\{\hat{r}_X[k]\right\}=\frac{1}{N}\sum_{\ell=k}^{N-1}E\left\{x[\ell]x^*[\ell-k]\right\}=\frac{1}{N}\sum_{\ell=k}^{N-1}r_X[k]=\frac{N-k}{N}r_X[k], \qquad k=-(N-1),\ldots,N-1$$

which may be expressed as

$$E\left\{\hat{r}_X[k]\right\}=w_B[k]r_X[k], \qquad w_B[k]=\begin{cases}\dfrac{N-|k|}{N}; & |k|\le N\\ 0; & |k|>N\end{cases}$$

where $w_B[k]$ is the Bartlett (triangular) window

spectrum estimation

if the rectangular window is used, the expected value of the periodogram is

$$E\left\{\hat{P}_{PER}\left(e^{j\omega}\right)\right\}=E\left\{\sum_{k=1-N}^{N-1}\hat{r}_X[k]e^{-jk\omega}\right\}=\sum_{k=1-N}^{N-1}E\left\{\hat{r}_X[k]\right\}e^{-jk\omega}=\sum_{k=-\infty}^{\infty}r_X[k]w_B[k]e^{-jk\omega}$$

using the frequency convolution theorem

$$E\left\{\hat{P}_{PER}\left(e^{j\omega}\right)\right\}=E\left\{\sum_{k=1-N}^{N-1}\hat{r}_X[k]e^{-jk\omega}\right\}=\frac{1}{2\pi}P_X\left(e^{j\omega}\right)*W_B\left(e^{j\omega}\right),\qquad \text{with}\quad W_B\left(e^{j\omega}\right)=\frac{1}{N}\left[\frac{\sin(N\omega/2)}{\sin(\omega/2)}\right]^2$$

Thus, the expected value of the periodogram is the convolution of the power spectrum $P_X(e^{j\omega})$ with the Fourier transform of the Bartlett window, and therefore, the periodogram is a biased estimate. However, since $W_B(e^{j\omega})$ converges to an impulse as N tends to infinity, the periodogram is asymptotically unbiased

$$\lim_{N\to\infty}E\left\{\hat{P}_{PER}\left(e^{j\omega}\right)\right\}=P_X\left(e^{j\omega}\right)$$

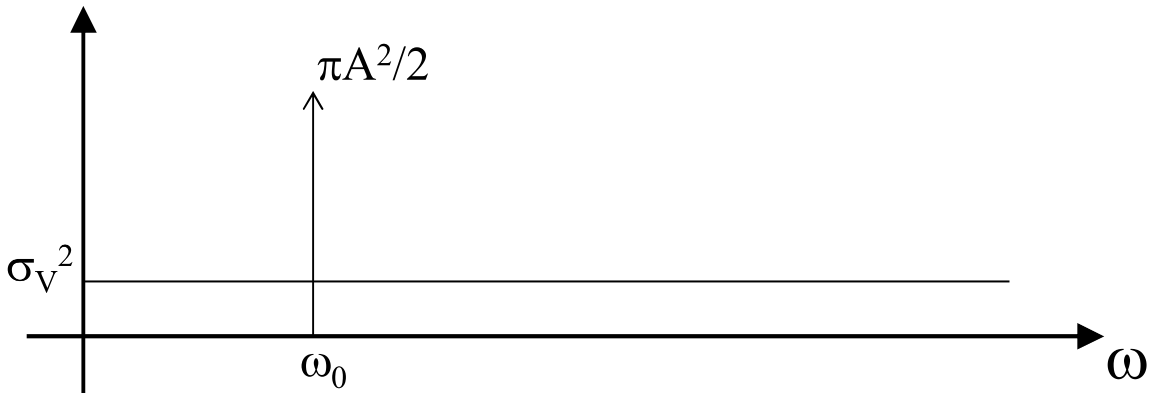
Note: it is equivalent to say «convolution of the power spectrum $P_X(e^{j\omega})$ with the Fourier transform of the Bartlett window of size 2N-1», or «convolution of the power spectrum $P_X(e^{j\omega})$ with the square of the Fourier transform of the rectangular window of size N»,

spectrum estimation

Example: a random process consists of a random-phase sinusoid in white noise with variance σ_V^2, find the power spectrum, consider the effect of the Bartlett window (the "lag window") and plot numerical results of the power spectrum estimation

Since $\boxed{x[n] = A\sin(n\omega_0 + \phi) + v[n]}$ and given that ϕ is a uniformly distributed random variable in the range $[-\pi, \pi]$, then the power spectrum results as

$$\boxed{P_X\left(e^{j\omega}\right) = \sigma_V^2 + \frac{\pi A^2}{2}\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]}$$

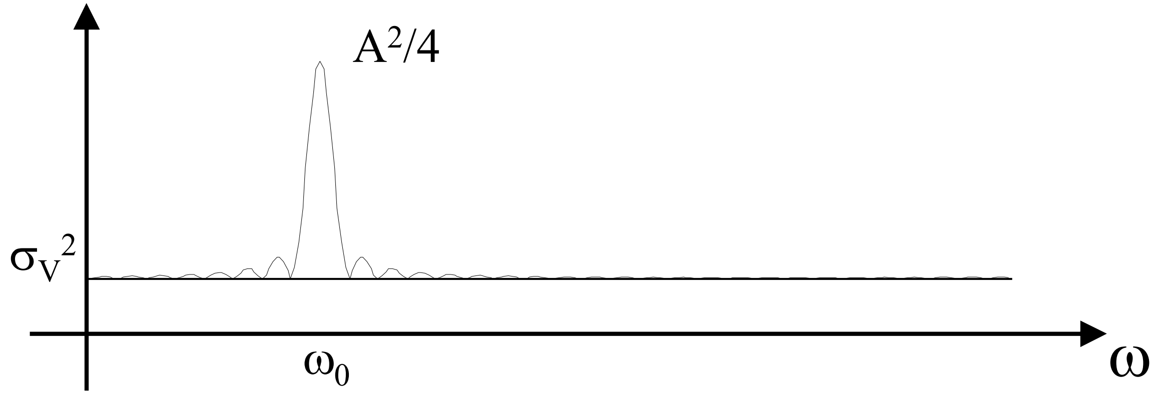


spectrum estimation

Example (cont.)

since $E\left\{\hat{P}_{PER}\left(e^{j\omega}\right)\right\}=\frac{1}{2\pi}P_X\left(e^{j\omega}\right)*W_B\left(e^{j\omega}\right)$ and $W_B\left(e^{j\omega}\right)=\frac{1}{N}\left[\frac{\sin\left(N\omega/2\right)}{\sin\left(\omega/2\right)}\right]^2$ then:

$$P_X\left(e^{j\omega}\right)=\sigma_V^2+\frac{A^2}{4}\left[W_B\left(e^{j(\omega-\omega_0)}\right)+W_B\left(e^{j(\omega+\omega_0)}\right)\right]$$



important consequences:

- a smoothing effect is noticeable that is due to $W_B(e^{j\omega})$ and that leads to a spreading of the power of the sinusoid over the main lobe of $|W_B(e^{j\omega})|$
- a power leakage through the sidelobes of the window, which creates secondary spectral peaks that may mask low-level components of the signal, and limits the ability of the periodogram to resolve closely-spaced frequencies
- the Bartlett window limits the ability of the periodogram to resolve closely-spaced narrow-band components in x[n], the resolution is about $0.89 \cdot 2\pi/N$

spectrum estimation

We recall that in order for the spectrogram to be a consistent estimator, the variance should vanish to zero when N tends to infinity

Monson Hayes shows (page 404) that while it is difficult to evaluate the variance of the spectrogram for a general random process, results may be found for white Gaussian noise, specifically M. Hayes shows that the covariance and the variance of the spectrogram are respectively given by:

$$\mathrm{Cov}\left\{\hat{P}_{PER}\left(e^{j\omega_1}\right)\hat{P}_{PER}\left(e^{j\omega 2}\right)\right\} = \sigma_x^4 \left[\frac{\sin N\left(\omega_1-\omega_2\right)/2}{N\sin\left(\omega_1-\omega_2\right)/2}\right]^2$$

and

$$\mathrm{V}ar\left\{\hat{P}_{PER}\left(e^{j\omega}\right)\right\} = \sigma_X^4 = P_X^2\left(e^{j\omega}\right)$$

which reveals that the variance does not converge to zero as N tends to infinity and, therefore, the periodogram is not a consistent estimator of the power spectrum (in the specific case it corresponds to white Gaussian noise)

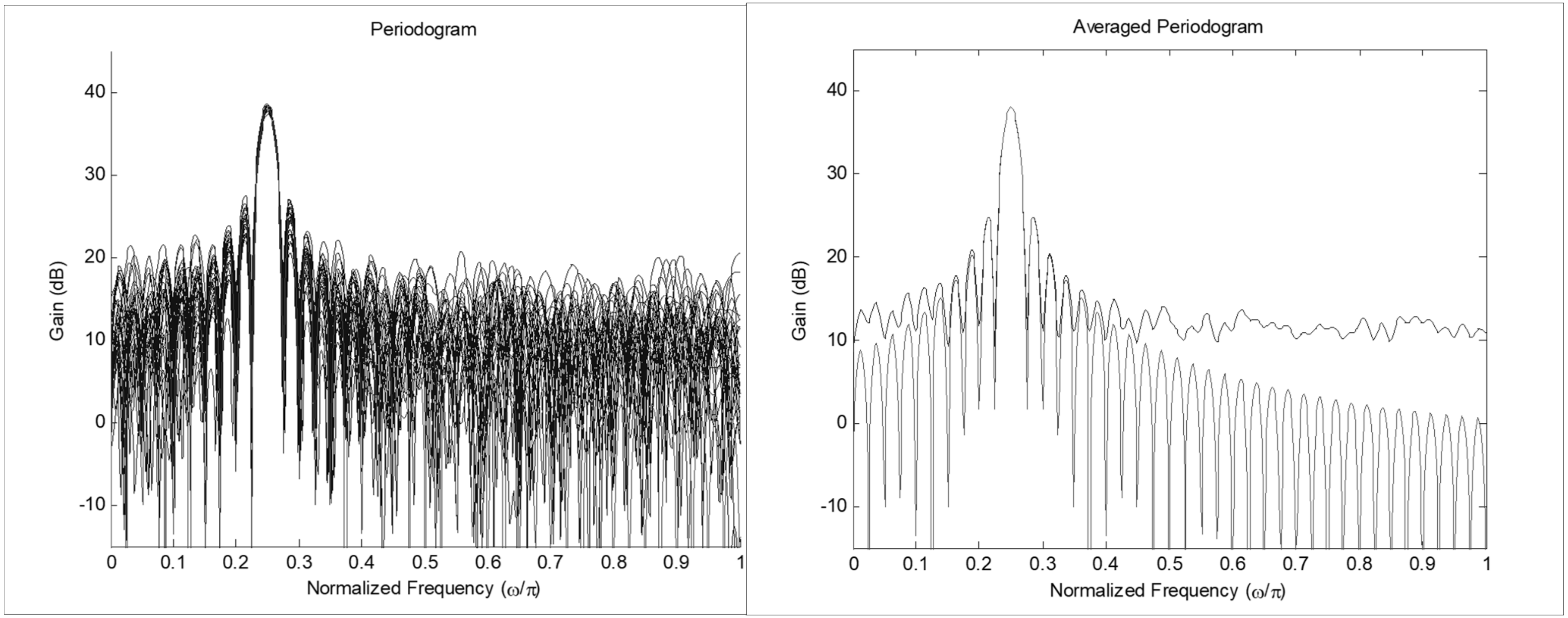
in the more general case of a Gaussian random process, it can be shown (M. Hayes, page 407) that the variance of the periodogram is proportional to the square of the power spectrum

spectrum estimation

– Matlab code illustrating spectrum estimation using the periodogram and the averaged spectrogram

```
A=2; N=80; % N is the length of the data vector
n=[0:N-1]; nptsfft=1024; % nptsfft is the FFT size
indx=[0:nptsfft/2]/(nptsfft/2); % frequency index up to Nyquist frequency
Psum=zeros(1,nptsfft/2+1); nsum=0; % for the averaged periodogram
omega=pi/4; % frequency of the sinusoid

for k=1:30   % try just 30 instances of the periodogram
   phi=2*(rand-0.5)*pi;
   x=A*sin(n*omega+phi)+0.4*randn(1,N);
   corr=xcorr(x); % this is the biased estimator
   X=fft(corr, nptsfft);
   P=(abs(X(1:nptsfft/2+1)));
   % this is equivalent to X=fft(x, nptsfft); P=(abs(X(1:nptsfft/2+1)).^2);
   figure(2); hold on;
   plot(indx, 10*log10(P)); hold off
   xlabel('Normalized Frequency (\omega/\pi)'); ylabel('Gain (dB)');
   title('Periodogram');

   Psum=Psum+P; nsum=nsum+1;
   figure(3); plot(indx, 10*log10(Psum/nsum)); axis([0 1 -15 45]);
   xlabel('Normalized Frequency (\omega/\pi)'); ylabel('Gain (dB)'); title('Averaged
   Periodogram')
end

% reference  Dirichlet function (periodic sinc function)
w=2*[0:nptsfft-1]/nptsfft-omega/pi; H=N*abs(sinc(w*N/2)./sinc(w/2));
figure(2); hold on; plot(indx, 10*log10((H(1:nptsfft/2+1)).^2),'m')
axis([0 1 -15 45]); hold off;

figure(3); hold on;
plot(indx, 10*log10((H(1:nptsfft/2+1)).^2),'m')
hold off
```

– figures generated by the above Matlab code



– the figure on the left illustrates an overlay of 30 periodograms and well as the true periodogram (magenta plot), the figure on the right represents the averaged spectrogram as well as the true periodogram (magenta plot),

– these figures illustrate that there is a considerable amount of variation from one single periodogram to the next and that averaging periodograms improves the convergence to the true power spectrum

spectrum estimation

- *the modified periodogram*

it has been shown previously that the periodogram is proportional to the squared magnitude of the Fourier transform of the windowed signal

$$\hat{P}_{PER}\left(e^{j\omega}\right)=\frac{1}{N}\left|X_W\left(e^{j\omega}\right)\right|^2=\frac{1}{N}\left|\sum_{k=-\infty}^{\infty}x_W[k]e^{-jk\omega}\right|^2=\frac{1}{N}\left|\sum_{k=-\infty}^{\infty}w[k]x[n]e^{-jk\omega}\right|^2$$

this result can be generalized for any window w[n] which leads to the modified periodogram; it can be shown that (presuming the window is real)

$$\hat{P}_{PER}\left(e^{j\omega}\right)=\frac{1}{N}\left|X_W\left(e^{j\omega}\right)\right|^2=\frac{1}{N}\sum_{k=-\infty}^{\infty}r_X[k]\left[\sum_{n=-\infty}^{\infty}w[n]w[n-k]\right]e^{-jk\omega}=\frac{1}{2\pi N}P_X\left(e^{j\omega}\right)*\left|W\left(e^{j\omega}\right)\right|^2$$

in particular, in the case of the rectangular window (w[n]=1, n=0,1,...,N-1), then

$$W_R\left(e^{j\omega}\right)=\frac{\sin\left(N\omega/2\right)}{\sin\left(\omega/2\right)}e^{-j(N-1)\omega/2}$$

Note: recall the Note in slide 10

- when compared to other windows, the rectangular window has the narrowest main lobe in the frequency response, but because it has high side lobes, it suffers from the strongest leakage effect (which means it may easily mask weak narrowband components)

spectrum estimation

The periodogram of a process that is windowed with a general window w[n] is called a modified periodogram and is given by

$$\hat{P}_M\left(e^{j\omega}\right)=\frac{1}{NU}\left|\sum_{k=-\infty}^{\infty}x[k]w[k]e^{-jk\omega}\right|^2$$

where N is the length of the window and U is a normalizing factor obtained as

$$U=\frac{1}{N}\sum_{k=-\infty}^{\infty}\left|w[k]\right|^2=\frac{1}{N}\sum_{k=0}^{N-1}\left|w[k]\right|^2=\frac{1}{2\pi N}\int_{-\pi}^{\pi}\left|W\left(e^{j\omega}\right)\right|^2 d\omega$$

Note: Munson Hayes provides (page 410) a Matlab function allowing to compute the periodogram

and its purpose is to make $P_M(e^{j\omega})$ asymptotically unbiased, in fact, in can be shown (M. Hayes, page 411) that when N tends to infinity

$$E\left\{\hat{P}_M\left(e^{j\omega}\right)\right\}=\frac{1}{2\pi NU}P_X\left(e^{j\omega}\right)*\left|W\left(e^{j\omega}\right)\right|^2\approx P_X\left(e^{j\omega}\right)$$

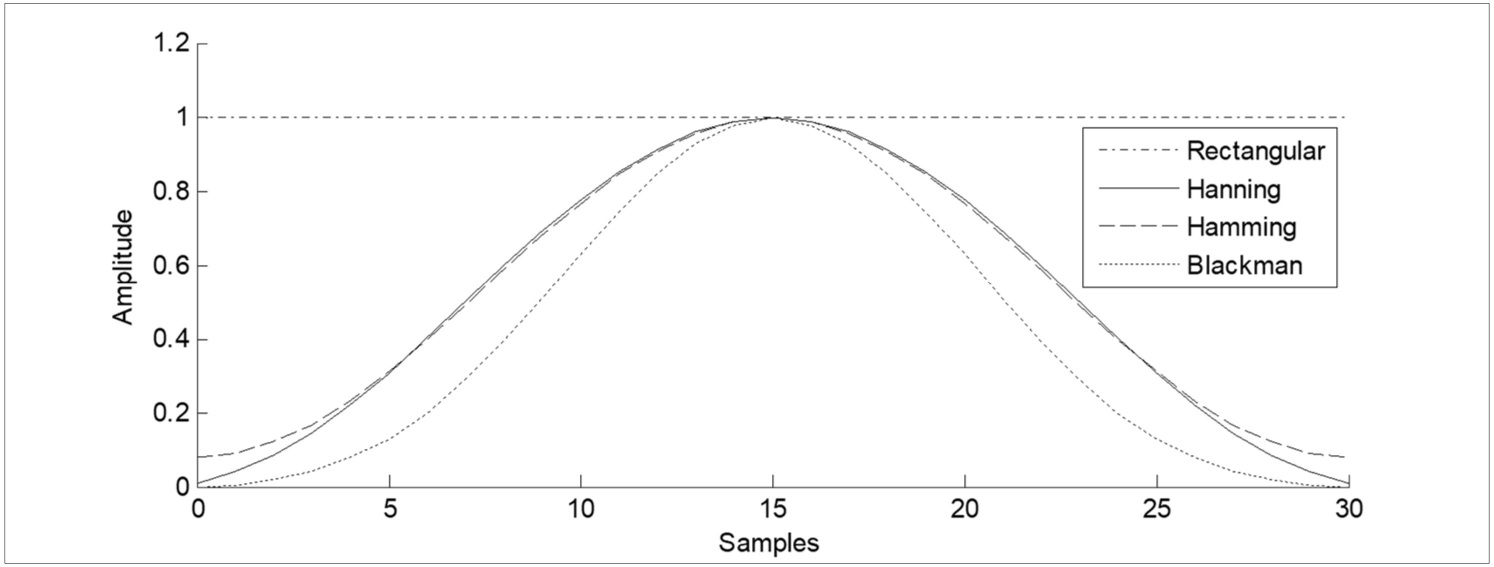
On the other hand, the variance of the modified periodogram is proportional to the square of the power spectrum (as the periodogram) which means the modified periodogram is not a consistent estimator of the power spectrum

- the window essentially provides a trade-off between spectral resolution (main lobe width) and spectral masking (sidelobe magnitude)

spectrum estimation

A few commonly used windows: Rectangular, Hanning, Hamming, Blackman

- impulse responses (N=31)



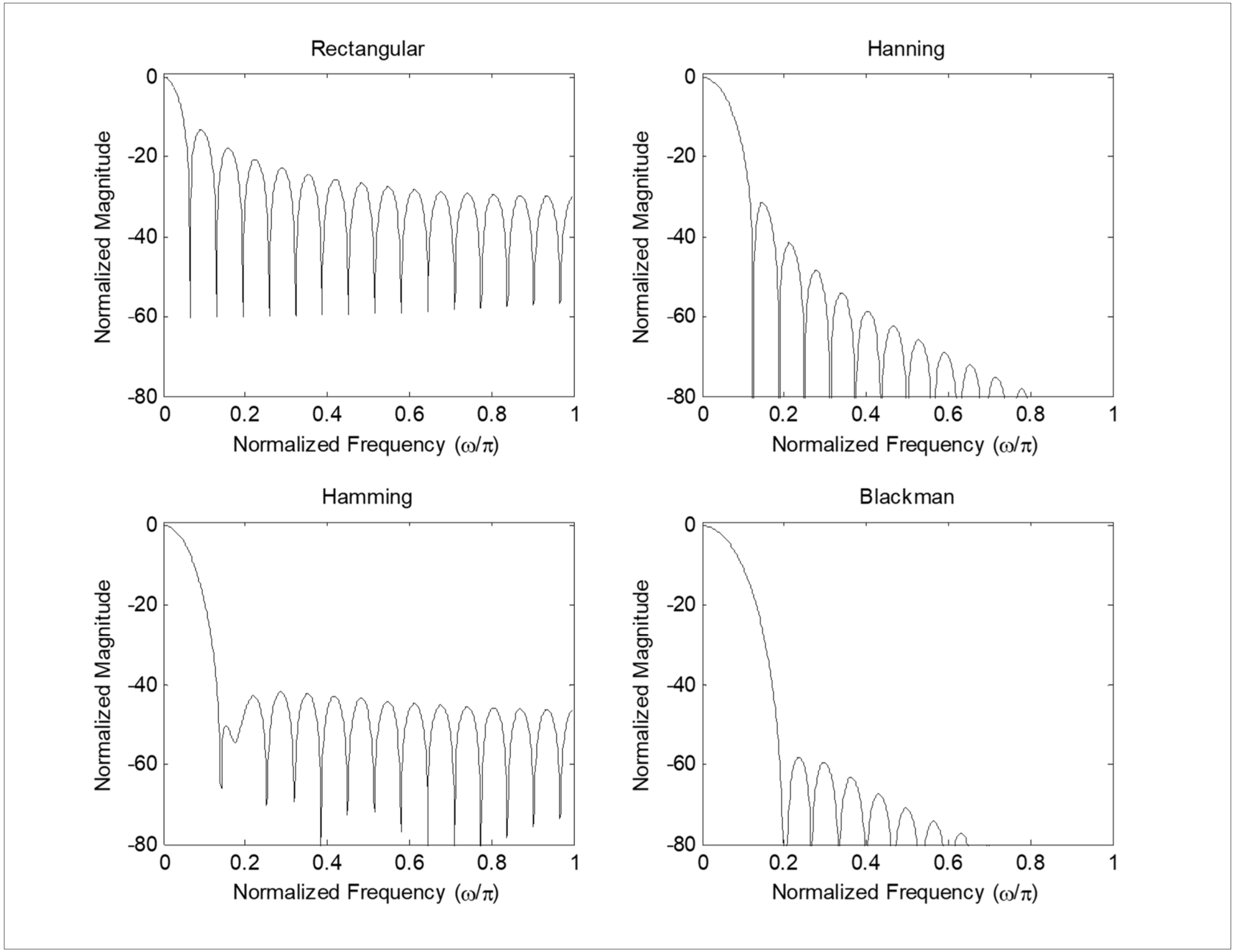
- windows are frequently compared using such properties as the main-side lobe attenuation and the 3dB bandwidth of the main lobe

window	sidelobe level (dB)	3 dB BW
Rectangular	-13	$0.89(2\pi/N)$
Hanning	-32	$1.44(2\pi/N)$
Hamming	-43	$1.30(2\pi/N)$
Blackman	-58	$1.68(2\pi/N)$

spectrum estimation

A few commonly used windows: Rectangular, Hanning, Hamming, Blackman

- frequency responses (N=31)



- *Bartlett's method: periodogram averaging*
  - unlike the periodogram or the modified periodogram, Bartlett's method is a consistent estimator of the power spectrum
    - since averaging a set of uncorrelated measurements of a random variable yields a consistent estimate of the mean, the main idea is therefore to estimate the power spectrum by periodogram averaging

The Bartlett estimate is given by $$\hat{P}_B\left(e^{j\omega}\right)=\frac{1}{LN}\sum_{\ell=0}^{L-1}\left|\sum_{n=0}^{N-1}x[n+\ell N]e^{-jn\omega}\right|^2$$

- the x[n] sequence is partitioned into L nonoverlapping sub-sequences of length N, a periodogram is obtained for each sub-sequence and a final power spectrum estimate is computed by averaging the L spectrograms

the expected value and variance of the Bartlett estimate are given respectively by

$$E\left\{\hat{P}_B\left(e^{j\omega}\right)\right\}=\frac{1}{2\pi}P_X\left(e^{j\omega}\right)*W_B\left(e^{j\omega}\right)$$ and $$\mathrm{V}ar\left\{\hat{P}_B\left(e^{j\omega}\right)\right\}\approx\frac{1}{L}P_X^2\left(e^{j\omega}\right)$$

if both L and N are allowed to tend to infinity, then the Bartlett method is a consistent estimator of the power spectrum

spectrum estimation

- *Welch's method: averaging modified periodograms*
  - consists of two modifications to Bartlett's method:
    - the sub-sequences of data are allowed to overlap
    - the sub-sequences of data are windowed

thus, Welch's method involves averaging a set of modified periodograms:

$$\hat{P}_{WELCH}\left(e^{j\omega}\right) = \frac{1}{LNU}\sum_{\ell=0}^{L-1}\left|\sum_{n=0}^{N-1} w[n]x[n+\ell D]e^{-jn\omega}\right|^2$$

(amount of overlap is N-D) the expected value of Welch's estimate is

$$E\left\{\hat{P}_{WELCH}\left(e^{j\omega}\right)\right\} = \frac{1}{2\pi NU} P_X\left(e^{j\omega}\right) * \left|W\left(e^{j\omega}\right)\right|^2$$

- which shows that as the previous periodogram-based estimators, Welch's method is an asymptotically unbiased estimator of the power spectrum

concerning variance, because the overlapping sub-sequences are not uncorrelated, evaluating the variance of estimate is difficult, a specific result has been found for 50% overlap and the Bartlett window:

$$\mathrm{V}ar\left\{\hat{P}_{WELCH}\left(e^{j\omega}\right)\right\} \approx \frac{9}{8L} P_X^2\left(e^{j\omega}\right)$$

Note: this variance is larger by a factor of 9/8 than the variance of the Bartlett estimator (M. Hayes, pag 418)

spectrum estimation

- *Blackman-Tukey method: periodogram smoothing*
  - whereas the previous methods aim at reducing the variance of the periodogram using averaging, this BT method aims at decreasing the statistical variability of the periodogram by de-emphasizing the contribution of unreliable estimates of the autocorrelation sequence to the periodogram
    - in fact, for finite data records of length N, the variance of $r_X[k]$ will be large when k is close N-1 (e.g. $r_X[N-1]=x[N-1]x[0]/N$), therefore these estimates are unreliable
    - one way to reduce the contribution of these estimates to the periodogram, is by applying a window to $r_X[k]$

  the Blackman-Tukey spectrum estimate is $$\hat{P}_{BT}\left(e^{j\omega}\right)=\sum_{k=-M}^{M}\hat{r}_X[k]w[k]e^{-jk\omega}$$

    - w[n] is the lag window, it has a double purpose: i) to limit the autocorrelation sequence such as to ignore the coefficients that have large variance, ii) to de-emphasize the contribution of $r_X[k]$ when k is close to M, a recommendation is that M=N/5
    - the lag window w[n] should be conjugate symmetric so that $W(e^{j\omega})$ is real-valued, also $W(e^{j\omega})$ should be positive such that $P_{BT}(e^{j\omega})$ is non-negative

  M. Hayes shows (pages 424-425) that periodogram-based power spectrum estimators have a performance that is quite comparable (and that depends essentially on the amount of data that is available), the main difference lies in the tradeoff between spectral resolution, leakage, and variance

spectrum estimation

- *periodogram-based frequency estimation*
  - the periodogram may be used to estimate the frequency (and also magnitude and phase) of a sinusoid while taking advantage of the knowledge of the window function, in our discussion here we will focus on frequency estimation of a sinusoid and using the rectangular window

The periodogram-based approach to frequency analysis may be represented by the sequence of processing steps:

$$x[n] \rightarrow x_W[n] = w[n]x[n] \xrightarrow{DFT} X_W[k] \rightarrow \left|X_W[k]\right| \rightarrow \omega_0$$

Let us assume x[n] is (as considered in slide 11) $x[n] = A\sin(n\omega_0 + \phi) + v[n]$ where v[n] represents white noise with variance σ_V^2, A represents the magnitude of the sinusoid and ϕ is a uniformly distributed random variable in the range $[-\pi, \pi]$. To simplify the analysis, we ignore the phase and noise as they do not have na influence of the main result we are looking for, therefore

$$x[n] \xleftrightarrow{F} X\left(e^{j\omega}\right) = \frac{A\pi}{j}\left[\delta\left(\omega - \omega_0\right) - \delta\left(\omega + \omega_0\right)\right]$$

spectrum estimation

and since the frequency response of the rectangular window is given by

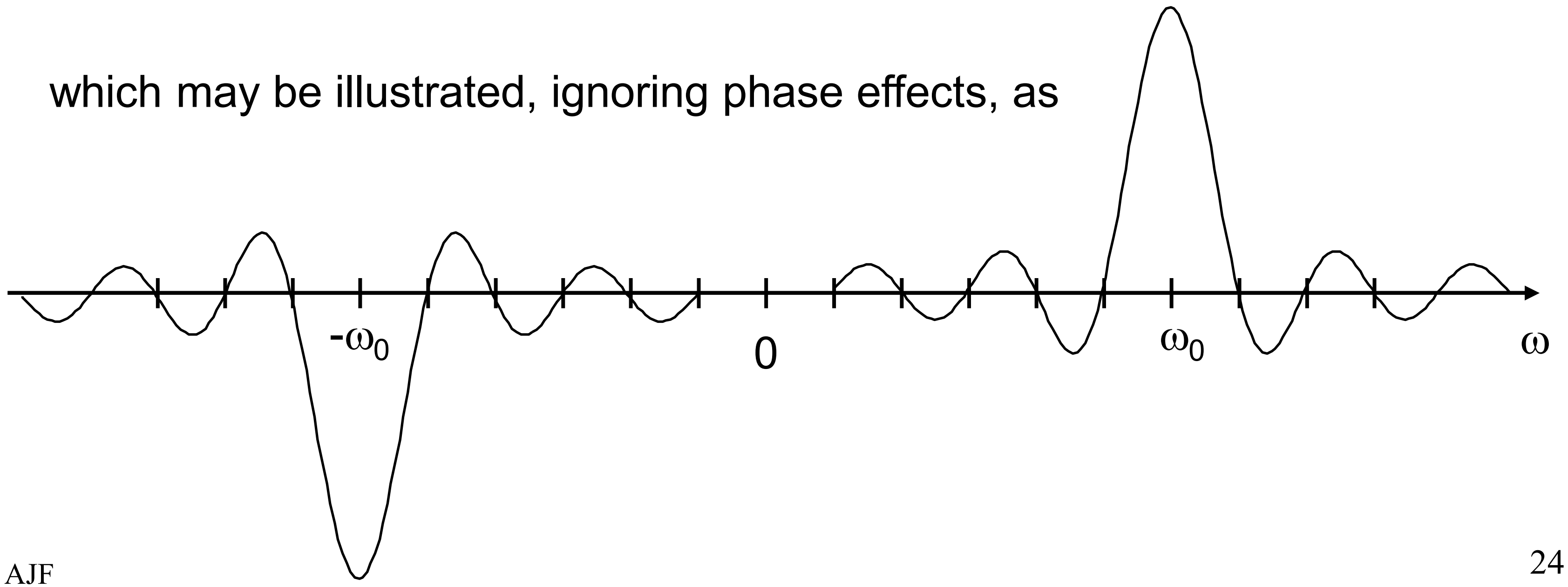
$$W\left(e^{j\omega}\right)=\sum_{n=0}^{N-1} w(n)e^{-j\omega n}=e^{-j\frac{N-1}{2}\omega}\,\frac{\sin\left(\omega\frac{N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}$$

then

$$x[n]w[n]\xleftrightarrow{F}\frac{1}{2\pi}X\left(e^{j\omega}\right)*W\left(e^{j\omega}\right)=X_W\left(e^{j\omega}\right)$$

$$X_W\left(e^{j\omega}\right)=e^{-j\frac{N-1}{2}\omega}\,\frac{A}{2j}\left[\frac{\sin\left(\omega-\omega_0\right)\frac{N}{2}}{\sin\frac{\left(\omega-\omega_0\right)}{2}}e^{j\frac{N-1}{2}\omega_0}-\frac{\sin\left(\omega+\omega_0\right)\frac{N}{2}}{\sin\frac{\left(\omega+\omega_0\right)}{2}}e^{-j\frac{N-1}{2}\omega_0}\right]$$

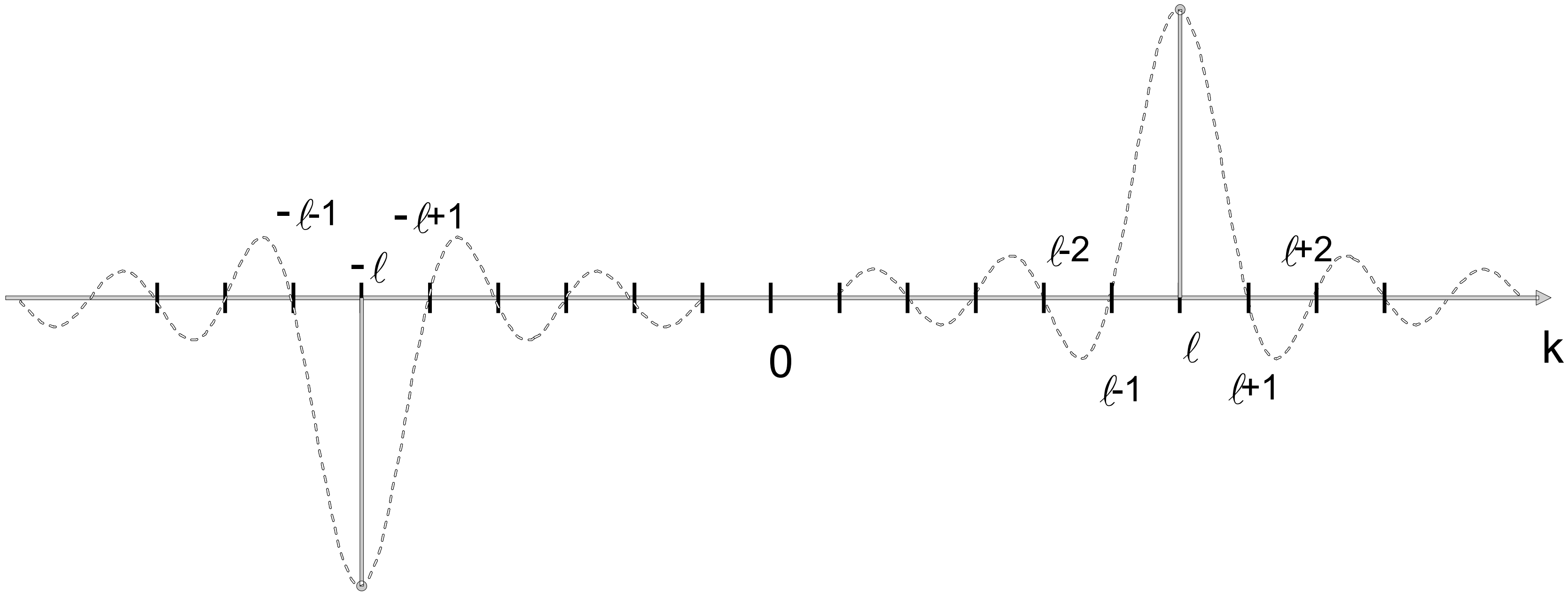
which may be illustrated, ignoring phase effects, as



spectrum estimation

As the DFT is a sampling of $X_W(e^{j\omega})$: $X_W[k] = X_W\left(e^{j\omega}\right)\Big|_{\omega = k2\pi/N,\quad k=0,1,\ldots,N-1}$

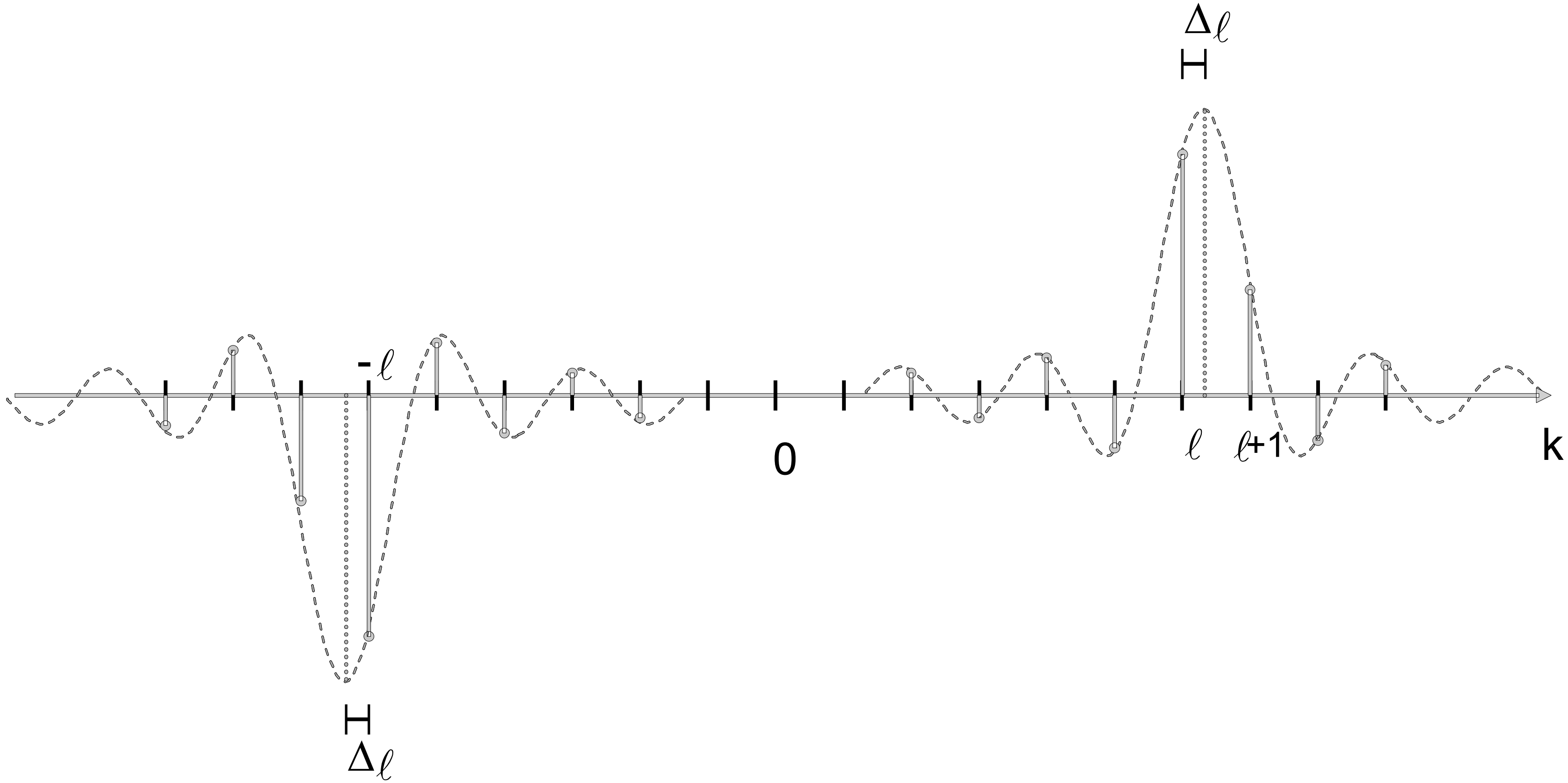
two situations may occur, either ω_0 corresponds to a frequency which is exactly sampled by the DFT (which is quite unlikely), in this case $\omega_0 = \frac{2\pi}{N}\ell$ where ℓ is an integer, and the sampling of $X_W(e^{j\omega})$ can be ilustrated as



or, in general, ω_0 is given by $\omega_0 = \frac{2\pi}{N}\left(\ell + \Delta\ell\right)$, where ℓ is an integer and $\Delta\ell$ is a real number, $0.0 \le \Delta\ell < 1.0$

spectrum estimation

In this case, leakage will occur, as expected, which can be illustrated as



which highlights that, at most, two spectral lines will fall within the main lobe of each one of the two periodic *sinc* functions, which correspond to the largest values in the magnitude spectrum $|X_W[k]|$. This suggests that since the exact shape of each sinc function is known and since the largest $|X_W[k]|$ values can be readily identified, it should be possible to estimate $\Delta\ell$.

spectrum estimation

In our analysis we consider only the magnitude spectrum $|X_W[k]|$ in the positive part of the frequency axis. Therefore

$$\left|X_W[k]\right| = \frac{A}{2}\left|\frac{\sin\left(k\frac{2\pi}{N}-\omega_0\right)\frac{N}{2}}{\sin\frac{\left(k\frac{2\pi}{N}-\omega_0\right)}{2}}\right|$$

and, considering that $\omega_0 = \frac{2\pi}{N}\left(\ell+\Delta\ell\right)$

the sampled periodic *sinc* function reduces to $\left|X_W[k]\right| = \frac{A}{2}\left|\frac{\sin k\pi(k-\ell-\Delta\ell)}{\sin k\pi(k-\ell-\Delta\ell)/N}\right|$

As a result, the two largest values in the magnitude spectrum (which form a local maximum in the magnitude spectrum) corrrespond to

$$\left|X_W[\ell]\right| = \frac{A}{2}\left|\frac{\sin k\pi(-\Delta\ell)}{\sin k\pi(-\Delta\ell)/N}\right| \quad \text{and} \quad \left|X_W[\ell+1]\right| = \frac{A}{2}\left|\frac{\sin k\pi(1-\Delta\ell)}{\sin k\pi(1-\Delta\ell)/N}\right|$$

By forming the ratio $|X_W[\ell]| / |X_W[\ell+1]|$ independence is achieved with respect to the sinusoid magnitude and gain of the window

spectrum estimation

Thus, is can be easily concluded that

$$\boxed{\frac{|X_W[\ell]|}{|X_W[\ell+1]|} = \sin\frac{\pi}{N}\cot\frac{\pi}{N}\Delta\ell - \cos\frac{\pi}{N}}$$

and, finally,

$$\boxed{\Delta\ell = \frac{N}{\pi}\arctan\frac{\sin\frac{\pi}{N}}{\cos\frac{\pi}{N} + \frac{|X_W[\ell]|}{|X_W[\ell+1]|}}}$$

It should be noted that this function is only valid if the rectangular window is used. It leads to the exact $\Delta\ell$ value in the case of a complex sinusoid (why ?) and in the absence of noise. For real sinusoids, or when multiple sinusoids exist, and under the influence of noise, estimates are only approximate. Still, Results are much better that coarse frequency estimation.

In fact, the maximum relative error of coarse frequency frequency estimation (which just says that the frequency of the sinusoid is obtained from the k index of the largest $|X_W[\ell]|$ DFT as $\ell 2\pi/N$) is $0.5*2\pi/N$ / ($2\pi/N$), i.e. 50% the frequency resolution of the DFT (also referred to as "bin width").

Simulation tests with a real sinusoid, N=128 and 10dB SNR, reveal that most frequently the relative estimation error is just about 3% of the bin width and, in the worst cases (when ℓ is close to 0 or to N/2-1), it can reach 10%.

spectrum estimation

The following Matlab code illustrates frequency estimation of a real sinusoid under 10 SNR (white, Gaussian) noise influence. ℓ and $\Delta\ell$ may be set as desired.

```
N=128; N2=N/2; n=[0:N-1]; trueell=13; truedeltaell=0.35;
omega=2*pi*(trueell+truedeltaell)/N;
A=1; SNR=10; sigmanoise=sqrt(A^2/(2*10^(SNR/10))); % standard deviation
h=rectwin(N).';
niter=100; relerror=0; accdata=[]; % run 100 iterations to average noise effects
for k=1:niter
    phi=2*(rand-0.5)*pi;
    x=A*sin(n*omega+phi); %  power is A^2/2 ~ sum(x.^2)/N
    noise=sigmanoise*randn(1,N); % power is sum(noise.^2)/N
%     10*log10((x*x.')/(noise*noise.')) % = 10*log10(var(x)/var(noise)) % check SNR
    x=x+noise;
    x=x.*h; X=fft(x); [value ell]=max(abs(X));
    if (ell==1 || ell==N2+1 || ell==N2+2 || ell==N) disp('No sinusoids found');
       return; end;
    if (abs(X(ell+1))<abs(X(ell-1))) ell=ell-1; end
    deltaell=N/pi*atan((sin(pi/N)/(cos(pi/N)+abs(X(ell)/X(ell+1))))); %freq estimate
    estomega=2*pi*(ell-1+deltaell)/N; % -1 is because of indexing in Matlab
    relerror=relerror+abs(estomega-omega)/(2*pi/N); % normalizes by DFT resolution
    accdata=[accdata abs(estomega-omega)/(2*pi/N)];
end
relerror=relerror/niter*100
figure(1); stem(n,abs(X)); figure(2); plot(accdata*100)
```

spectrum estimation

- *useful Matlab functions*

[Pxx,W] = PERIODOGRAM(X,WINDOW,NFFT)

- Power Spectral Density (PSD) estimate via periodogram method

[Pxx,W] = PWELCH(X,WINDOW,NOVERLAP,NFFT)

- Power Spectral Density estimate via Welch's method

rectwin, bartlett, blackman, chebwin, hamming, hann, hanning, kaiser,…

- window functions

H = SPECTRUM.<ESTIMATOR>

- SPECTRUM Spectral Estimation
- returns a spectral estimator in H of type specified by ESTIMATOR
- There are three types of estimators: power spectral density (PSD), mean-square spectrum (MSS), and pseudo spectrum

spectrum estimation

- *Minimum variance (MV) spectrum estimation*
  - the power spectrum is estimated by filtering a process with a bank of narrowband bandpass filters, which is similar to the interpretation given in slides 6, 7 and 8 to the spectrogram, the difference lies however in the fact that the filters are not simple modulations (to a frequency that is multiple of $2\pi/N$) of a baseband prototype filter, in fact in MV spectrum estimation, the filters are data dependent
    - each filter in the filter bank is data adaptive and "optimum" in the sense of rejecting as much as possible out-of-band signal power

MV spectrum estimation consists of three steps:

1. design a bank of band-pass filters $g_\ell[n]$ with center frequency ω_ℓ and effectively rejecting out of band power
2. estimate the power in each output $y_\ell[n]$
3. set $P_X(e^{j\omega})$ equal to the power estimated in 2 divided by the filter bandwidth

if $g_\ell[n]$ is a complex-valued FIR filter of order p, in order not to modify the power of the signal at frequency ω_ℓ

$$G_\ell\left(e^{j\omega_\ell}\right)=\sum_{n=0}^{p} g_\ell[n]e^{-jn\omega_\ell}=1$$

spectrum estimation

if $\mathbf{g}_\ell = \left[g_\ell[0], \quad g_\ell[1], \ldots \quad g_\ell[p]\right]^T$ and $\mathbf{e}_\ell = \left[1, \quad e^{j\omega_\ell}, \ldots \quad e^{jp\omega_\ell}\right]^T$ then

$$\mathbf{e}_\ell^H \mathbf{g}_\ell = 1 = (1)^H = \left(\mathbf{e}_\ell^H \mathbf{g}_\ell\right)^H = \mathbf{g}_\ell^H \mathbf{e}_\ell$$

and given that (week 3, slide 6)

$$r_x[n] \longrightarrow \boxed{h[n]} \xrightarrow{\; r_{yx}[n] \;} \boxed{h^*[-n]} \longrightarrow r_y[n]$$

$$r_y[n] = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k]\, r_x[m-k]\, h^*[m-n]$$

$$E\left\{|y[n]|^2\right\} = \sigma_Y^2 = r_y[0] = \sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h[k]\, r_x[m-k]\, h^*[m] = \mathbf{h}^H \mathbf{R}_X \mathbf{h}$$

which, in our case leads to $E\left\{|y_\ell[n]|^2\right\} = \mathbf{g}_\ell^H \mathbf{R}_X \mathbf{g}_\ell$, minimizing this result subject to $\mathbf{e}_\ell^H \mathbf{g}_\ell = 1$ leads to (M.Hayes, section 2.3.10):

$$\mathbf{g}_\ell = \frac{\mathbf{R}_X^{-1}\mathbf{e}_\ell}{\mathbf{e}_\ell^H \mathbf{R}_X^{-1} \mathbf{e}_\ell} \qquad \text{and} \qquad \min_{\mathbf{g}_\ell} E\left\{|y_\ell[n]|^2\right\} = \frac{1}{\mathbf{e}_\ell^H \mathbf{R}_X^{-1} \mathbf{e}_\ell}$$

Note: although these results were derived for ω_ℓ they can be generalized for all ω

spectrum estimation

the MV power spectrum estimate is obtained by dividing the power estimate by the bandwidth of the bandpass filter (M. Hayes, page 427)

$$\hat{P}_{MV}\left(e^{j\omega}\right)=\frac{p+1}{\mathbf{e}^H\mathbf{R}_X^{-1}\mathbf{e}}$$

Note: Since R_X is Toeplitz, the inverse may be found using either the Levinson recursion or the Cholesky decomposition

- M.Hayes details (pages 427-433) ways to efficiently compute this result as well as to use it to estimate the frequency of a sinusoid or a complex exponential in noise
- since this result implies the inversion of the autocorrelation matrix R_X, in order to reduce computational costs, the filter order (p) and, therefore, the matrix order is generally much smaller than the data length (N)
  - also to avoid the large variance of the autocorrelation estimates for values of k that are close to N

Note: Munson Hayes provides (page 430) a Matlab function `minvar()` allowing to compute the MV spectrum

spectrum estimation

- *the Maximum Entropy method (MEM)*
  - this method is suited to estimate more accurately the power spectrum in those cases where the autocorrelation sequence needs to be extrapolated beyond the lag corresponding to the length of the data, N
    - classical methods extrapolate the autocorrelation sequence for lags $|k|>N$ with zeros, which is equivalent to windowing the autocorrelation sequence
    - many signals of interest (e.g. narrowband processes that have autocorrelations that decay slowly with *k*) have autocorrelations that are nonzero for $|k|\geq N$, in these cases windowing may significantly limit the resolution and accuracy of the estimated spectrum
    - MEM suggests one possible way to perform extrapolation

if the autocorrelation values $r_X[k]$ of a WSS process are known up to lag $|k|\leq p$, and $r_E[k]$ are extrapolated, then the power spectrum is

$$P_X\left(e^{j\omega}\right)=\sum_{k=-p}^{p} r_X[k]e^{-jk\omega}+\sum_{|k|>p} r_E[k]e^{-jk\omega}$$

and should correspond to a valid spectrum, i.e. $P_X(e^{j\omega})$ should be real-valued and non-negative for all ω

spectrum estimation

extrapolating $r_E[k]$ while complying with these constraints may be achieved in such a way as to maximize the entropy (a measure of randomness or uncertainty) of the process

- a maximum entropy criterion extrapolation is equivalent to finding the sequence of autocorrelations $r_E[k]$ that make x[n] as white as possible, i.e. such that $P_X(e^{j\omega})$ is as flat as possible

for a Gaussian random process with power spectrum $P_X(e^{j\omega})$ the entropy is

$$H(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \ln P_X\left(e^{j\omega}\right) d\omega$$

when only the partial autocorrelation sequence $r_X[k]$, $|k| \le p$, is known, the $r_E[k]$ values are found so that the entropy is maximized and subject to the condition that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_X\left(e^{j\omega}\right) e^{jk\omega} d\omega = r_X[k], \quad |k| \le p$$

spectrum estimation

the values of $r_E[k]$ that maximize the entropy may be found by setting

$$\frac{\partial H(x)}{\partial r_E^*[k]} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{P_X\left(e^{j\omega}\right)}\frac{\partial P_X\left(e^{j\omega}\right)}{\partial r_E^*[k]}d\omega = 0, \quad |k|>p$$

using the conjugate symmetry of $r_X[k]$ ($r_X[-k]=r_X^*[k]$) and (equation in slide 27)

$$P_X\left(e^{j\omega}\right)=\sum_{k=-p}^{p} r_X[k]e^{-jk\omega}+\sum_{|k|>p} r_E^*[k]e^{jk\omega}$$

then, one obtains $\frac{\partial H(x)}{\partial r_E^*[k]} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{P_X\left(e^{j\omega}\right)}e^{jk\omega}d\omega = 0, \quad |k|>p$ which means the inverse Fourier transform of the reciprocal of the power spectrum is a finite-length sequence that is zero for |k|>p, i.e. defining $Q_X\left(e^{j\omega}\right)=\frac{1}{P_X\left(e^{j\omega}\right)}$ then

$$q_X[k]=\frac{1}{2\pi}\int_{-\pi}^{\pi}Q_X\left(e^{j\omega}\right)e^{jk\omega}d\omega = 0, \quad |k|>p$$

spectrum estimation

therefore $Q_X\left(e^{j\omega}\right)=\dfrac{1}{P_X\left(e^{j\omega}\right)}=\sum_{k=-p}^{p} q_X[k]e^{-jk\omega}$ reveals that the maximum

entropy power spectrum for a Gaussian process, is an all-pole power spectrum

$$\hat{P}_{MEM}\left(e^{j\omega}\right)=\frac{1}{Q_X\left(e^{j\omega}\right)}=\frac{1}{\sum_{k=-p}^{p} q_X[k]e^{-jk\omega}}$$

and using the spectral factorization theorem, one obtains

$$\hat{P}_{MEM}\left(e^{j\omega}\right)=\frac{\left|b[0]\right|^2}{A_p\left(e^{j\omega}\right)A_p^*\left(e^{j\omega}\right)}=\frac{\left|b[0]\right|^2}{\left|1+\sum_{k=1}^{p} a_p[k]e^{-jk\omega}\right|^2}=\frac{\left|b[0]\right|^2}{\left|\mathbf{e}^H\mathbf{a}_p\right|^2}$$

which uses $\mathbf{a}_p=\left[1, \quad a_p[1], \ldots \quad a_p[p]\right]^T$ and $\mathbf{e}=\left[1, \quad e^{j\omega}, \ldots \quad e^{jp\omega}\right]^T$

Therefore, the $a_p[k]$ and $b[0]$ coefficients should be chosen in such a way that the inverse Fourier Transform of $P_X(e^{j\omega})$ produces an autocorrelation sequence that matches $r_X[k]$, $|k|\leq p$

spectrum estimation

This is equivalent to say that the $a_p[k]$ coefficients are the solution to the autocorrelation normal equations:

$$\begin{bmatrix} r_x[0] & r_x^*[1] & \cdots & r_x^*[p] \\ r_x[1] & r_x[0] & \cdots & r_x^*[p-1] \\ r_x[2] & r_x[1] & \cdots & r_x^*[p-2] \\ \vdots & \vdots & & \vdots \\ r_x[p] & r_x[p-1] & \cdots & r_x[0] \end{bmatrix} \begin{bmatrix} 1 \\ a_p[1] \\ a_p[2] \\ a_p[3] \\ \vdots \\ a_p[p] \end{bmatrix} = \varepsilon_p \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and, in addition, if $\left|b[0]\right|^2 = r_X[0] + \sum_{k=1}^{p} a_p[k] r_X^*[k] = \varepsilon_p$ then the autocorrelation

matching constraint is satisfied: $\frac{1}{2\pi} \int_{-\pi}^{\pi} P_X\left(e^{j\omega}\right) e^{jk\omega} d\omega = r_X[k], \quad |k| \le p$

Therefore: $\hat{P}_{MEM}\left(e^{j\omega}\right) = \frac{\varepsilon_p}{\left|\mathbf{e}^H \mathbf{a}_p\right|^2}$

spectrum estimation

In summary, given a sequence of autocorrelations $r_X[k]$, k=0,1,… p , the MEM spectrum is computed as follows

1. the coefficients $a_p[k]$ and ε are computed using the autocorrelation normal equations
2. the MEM spectrum is formed using these parameters into $$\hat{P}_{MEM}\left(e^{j\omega}\right)=\frac{\varepsilon_p}{\left|\mathbf{e}^H\mathbf{a}_p\right|^2}$$
3. since the MEM spectrum is an all-pole power spectrum, then $r_X[k]$ satisfies the Yule-Walker equations

$$r_X[\ell]=-\sum_{k=1}^{p}a_p[k]\,r_X[k-\ell]$$

which is a recursion allowing to extrapolate the autocorrelation sequence

Final note: The MEM method attempts to extrapolate the autocorrelation sequence while imposing the least amount of structure on the data (i.e. it performs a maximum entropy extrapolation), however since it imposes an all-pole model on the data, unless the process is known to be consistent with the model, the estimated spectrum may not be very accurate.

Note: Munson Hayes provides (page 437) a Matlab function `mem()` allowing to compute the ME spectrum

spectrum estimation

- *parametric methods*
  - are particularly important and advantageous when it is possible to incorporate a model for the process directly into the spectrum estimation
    - a more accurate and higher resolution estimate is likely to be found
  - the first step is to select an appropriate model for the process
    - methods that are commonly used include the autoregressive (AR), moving average (MA), autoregressive moving average (ARMA), and harmonic (complex exponentials in noise)
  - the second step is to find the model parameters from the data
  - the final step is to estimate the power spectrum by incorporating the estimated parameters into the parametric form for the spectrum
  - a word of caution: although it is possible to significantly improve the resolution of the spectrum estimate with a parametric model, unless the model that is used is appropriate for the process that is being analyzed, inaccurate or misleading estimates may be obtained

spectrum estimation

- *autoregressive spectrum estimation*
  - an autoregressive process, x[n], may be represented as the output of an all-pole filter that is driven by unit variance white noise, an estimate of the power spectrum is formed using the estimated parameters b[0] and a[k]

$$\hat{P}_{AR}\left(e^{j\omega}\right)=\frac{\left|\hat{b}[0]\right|^2}{\left|1+\sum_{k=1}^{p}\hat{a}_p[k]e^{-jk\omega}\right|^2}$$

  - the accuracy of $P_{AR}(e^{j\omega})$ depends on how accurately the model parameters may be estimated and, more importantly, on whether or not an autoregressive model is consistent with the way the data is generated

The Autocorrelation Method

- in the autocorrelation method of all-pole modeling, the AR coefficients are found by solving the the autocorrelation normal equations

$$\begin{bmatrix} r_x[0] & r_x^*[1] & \cdots & r_x^*[p] \\ r_x[1] & r_x[0] & \cdots & r_x^*[p-1] \\ r_x[2] & r_x[1] & \cdots & r_x^*[p-2] \\ \vdots & \vdots & & \vdots \\ r_x[p] & r_x[p-1] & \cdots & r_x[0] \end{bmatrix}\begin{bmatrix} 1 \\ a_p[1] \\ \vdots \\ a_p[p] \end{bmatrix}=\varepsilon_p\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $$\hat{r}_X[k] = \frac{1}{N}\sum_{\ell=k}^{N-1} x[\ell]\,x^*[\ell-k], \qquad k = 0,1,\ldots,p$$

after the $r_X[k]$ autocorrelations are found, then $$|b[0]|^2 = \varepsilon_p = r_X[0] + \sum_{k=1}^{p} a_p[k]\,r_X^*[k]$$

and $P_{AR}(e^{j\omega})$ can be obtained, this method is also known as the Yule-Walker method.

It should be noted that the Yule-Walker method (which assumes x[n] is an autoregressive process) is equivalent to the maximum entropy method (which assumes x[n] is Gaussian)

– since the autocorrelation matrix R is Toeplitz, the Levinson-Durbin recursion may be used to solve the normal equations
  - the autocorrelation method generally produces a low resolution estimate (when compared for example to the covariance method) because, according to this method, a rectangular window is applied to the data when estimating the autocorrelation sequence
  - an artifact known as *spectral line splitting* may occur when the autocorrelation method is used, it consists in a splitting of the of a single spectral peak into two separate and distinct peaks, this may be explained by overmodeling of x[n], i.e. when p is too large

- a variation of the above autocorrelation estimate (which is biased) is the unbiased version:

$$\hat{r}_X[k] = \frac{1}{N-k}\sum_{\ell=k}^{N-1} x[\ell]\,x^*[\ell-k], \qquad k = 0,1,\ldots,p$$

however, in this case the autocorrelation matrix is not guaranteed to be positive definite and, as a consequence, the variance of the spectrum estimate tends to be large

  - this explains that in general the biased estimate of $r_X[k]$ is preferred over the unbiased estimate

The Covariance Method

- the covariance method involves finding the solution to the set of linear equations (whose underlying matrix is not Toeplitz)

$$\begin{bmatrix} r_x[1,1] & r_x[1,2] & \cdots & r_x[1,p] \\ r_x[2,1] & r_x[2,2] & \cdots & r_x[1,p-1] \\ r_x[3,1] & r_x[3,2] & \cdots & r_x[1,p-2] \\ \vdots & \vdots & & \vdots \\ r_x[p,1] & r_x[p-1,2] & \cdots & r_x[p,p] \end{bmatrix} \begin{bmatrix} a_p[1] \\ a_p[2] \\ \vdots \\ a_p[p] \end{bmatrix} = - \begin{bmatrix} r_x[0,1] \\ r_x[0,2] \\ \vdots \\ r_x[0,p] \end{bmatrix}$$

spectrum estimation

where $$r_X[k,\ell]=\sum_{n=p}^{N-1} x[n-\ell]\,x^*[n-k]$$

- for short data records, the covariance method generally produces higher resolution spectrum estimates than the autocorrelation method, when N>>p the effect of windowing is reduced and the difference between the two approaches becomes negligible (M. Hayes describes in pages 444, 445 two variations of the covariance method, the modified covariance method and Burg's method which have a number of advantages, namely in terms of accuracy and stability)

Selecting the Model Order

- if the model order is too small, the resulting spectrum will be smoothed and will have a poor resolution
- if the model order is too large, then the spectrum may contain spurious peaks and may be affected by spectral line splitting

spectrum estimation

- *moving average spectrum estimation*
  - as already discussed before, a moving average process may be generated by filtering unit variance white noise, w[n], with an FIR filter

$$x[n]=\sum_{k=0}^{q} b_q[k]\,w[n-k]$$, the power spectrum results as $$P_X\left(e^{j\omega}\right)=\left|\sum_{k=0}^{q} b_q[k]\,e^{-jk\omega}\right|^2$$

equivalently, the power spectrum, may be obtained as a function of the

autocorrelation sequence $r_X[k]$: $$P_X\left(e^{j\omega}\right)=\sum_{k=-q}^{q} r_X[k]\,e^{-jk\omega}$$ where $r_X[k]$: is related

to the filter coefficients $b_q[k]$ through the Yule-Walker equations:

$$r_X[k]=\sum_{\ell=0}^{q-k} b_q[\ell+k]\,b_q^*[\ell] \quad ; \quad k=0,1,\ldots,q$$ therefore, given that $r_X[k]$, |k|>q, and

using a suitable estimate of the autocorrelation sequence, the power

spectrum estimate is obtained as $$\hat{P}_{MA}\left(e^{j\omega}\right)=\sum_{k=-q}^{q} \hat{r}_X[k]\,e^{-jk\omega}$$

Note: this is equivalent to the Blackman -Tukey estimate using the rectangular window

spectrum estimation

- *autoregressive moving average spectrum estimation*
  - as also discussed before, an autoregressive moving average process has a power spectrum of the form

$$P_X\left(e^{j\omega}\right)=\frac{\left|\sum_{k=0}^{q} b_q[k]e^{-jk\omega}\right|^2}{\left|1+\sum_{k=1}^{p} a_p[k]e^{-jk\omega}\right|^2}$$

, the filter

$$H(z)=\frac{B_q(z)}{A_p(z)}=\frac{\sum_{k=0}^{q} b_q[k]z^{-k}}{1+\sum_{k=1}^{p} a_p[k]z^{-k}}$$

having both poles and zeros, may be used to generate such power spectrum by filtering unit variance white noise, therefore the power spectrum of ARMA processes may be estimated using

$$\hat{P}_{ARMA}\left(e^{j\omega}\right)=\frac{\left|\sum_{k=0}^{q} \hat{b}_q[k]e^{-jk\omega}\right|^2}{\left|1+\sum_{k=1}^{p} \hat{a}_p[k]e^{-jk\omega}\right|^2}$$

where the AR parameters may be estimated from the modified Yule-Walker equations and the MA model parameters may be estimated (for example) using Durbin's method

spectrum estimation

- *frequency estimation*
  - many signals of interest (sonar, speech processing,…) have a structure of complex sinusoids in white noise (w[n]):

$$x[n] = \sum_{\ell=1}^{p} A_\ell \, e^{jn\omega_\ell} + w[n]$$

in general the amplitudes are complex, i.e. $A_\ell = |A_\ell| e^{j\phi_\ell}$ where ϕ_ℓ is uniformly distributed over the interval [-π, π]

the frequencies ω_ℓ and the $|A_\ell|$ are normally (not random but are) unknown

  - thus the power spectrum of x[n] consists of a set of *p* pulses of area $2\pi|A_\ell|$ at frequencies ω_ℓ for ℓ=1,2,…,p , plus the power spectrum of the white noise
  - in these cases it is the estimation of the amplitudes and frequencies that is of interest, rather than the overall power spectrum
  - we will discuss several methods of frequency estimation that are based on an eigendecomposition of the autocorrelation matrix into two subspaces: a signal subspace and a noise subspace

spectrum estimation

- *eigendecomposition of the autocorrelation matrix*
  - is an approach that may be used for frequency estimation

taking as an example $x[n] = A_1 e^{jn\omega_1} + w[n]$, where $A_1 = |A_1| e^{j\phi_1}$, ϕ_1 is a uniformly distributed random variable over the interval [-π, π], and w[n] is white noise that has variance σ_w^2 , then

$$r_X[\ell] = E\{x[n]x^*[n-\ell]\} = |A_1|^2 \, E\{e^{j\ell\omega_1}\} + E\{w[n]w^*[n-\ell]\} = |A_1|^2 \, e^{j\ell\omega_1} + \sigma_W^2 \delta[\ell]$$

if, to simplify, $P_1 = |A_1|^2$, the autocorrelation matrix $\mathbf{R}_X$ is $\mathbf{R}_X = \mathbf{R}_{SIG} + \mathbf{R}_{NOI}$ where the signal autocorrelation matrix $\mathbf{R}_{SIG}$ is

$$\mathbf{R}_{SIG} = P_1 \begin{bmatrix} 1 & e^{-j\omega_1} & \cdots & e^{-j(M-1)\omega_1} \\ e^{j\omega_1} & 1 & \cdots & e^{-j(M-2)\omega_1} \\ e^{j2\omega_1} & e^{j\omega_1} & \cdots & e^{-j(M-3)\omega_1} \\ \vdots & \vdots & & \vdots \\ e^{j(M-1)\omega_1} & e^{j(M-2)\omega_1} & \cdots & 1 \end{bmatrix}$$

and has rank one; the noise autocorrelation matrix is diagonal: $\mathbf{R}_{NOI} = \sigma_W^2 \mathbf{I}$

Defining $\mathbf{e}_1 = \left[1, \quad e^{j\omega_1}, \quad e^{j2\omega_1}, \ldots \quad , e^{j(M-1)\omega_1}\right]^T$ then $\mathbf{R}_{SIG}$ may be written in

terms of $\mathbf{e}_1$ as $\mathbf{R}_{SIG} = P_1 \mathbf{e}_1 \mathbf{e}_1^H$; since the rank of $\mathbf{R}_{SIG}$ is one, then $\mathbf{R}_{SIG}$ has only one non-zero eigenvalue which may be found using

$$\mathbf{R}_{SIG}\mathbf{e}_1 = P_1\left(\mathbf{e}_1\mathbf{e}_1^H\right)\mathbf{e}_1 = P_1\mathbf{e}_1\left(\mathbf{e}_1^H\mathbf{e}_1\right) = P_1\mathbf{e}_1 M = MP_1\mathbf{e}_1 = \lambda_1^{SIG}\mathbf{e}_1$$

the non-zero eigenvalue is therefore MP_1 and $\mathbf{e}_1$ is the corresponding eigenvector; on the other hand, since $\mathbf{R}_{SIG}$ is Hermitian, then the remaining eigenvectors $\mathbf{v}_2$, $\mathbf{v}_3$,…, $\mathbf{v}_M$ will be orthogonal to $\mathbf{e}_1$, i.e. $\mathbf{e}_1^H \mathbf{v}_k = 0, \quad k = 2,3,\ldots, M$

(recall that for a Hermitian matrix, the eigenvectors corresponding to distinct eigenvalues, are orthogonal)

if the eigenvalues of $\mathbf{R}_{SIG}$ are λ_i^{SIG}, then

$$\mathbf{R}_X\mathbf{v}_i = \left(\mathbf{R}_{SIG} + \sigma_W^2\mathbf{I}\right)\mathbf{v}_i = \left(\mathbf{R}_{SIG}\mathbf{v}_i + \sigma_W^2\mathbf{I}\mathbf{v}_i\right) = \left(\lambda_i^{SIG}\mathbf{v}_i + \sigma_W^2\mathbf{v}_i\right) = \left(\lambda_i^{SIG} + \sigma_W^2\right)\mathbf{v}_i$$

i.e. the eigenvectors of $\mathbf{R}_{SIG}$ are also the eigenvectors of $\mathbf{R}_X$, and the eigenvalues of $\mathbf{R}_X$ are

$$\lambda_i^{SIG} + \sigma_W^2$$

spectrum estimation

As a result, the largest eigenvalue of $\mathbf{R}_{SIG}$ is $\boxed{\lambda_{\max}^{SIG} = \lambda_1^{SIG} + \sigma_W^2 = MP_1 + \sigma_W^2}$ and the remaining M-1 eigenvalues are equal to σ_w^2. Thus, it is possible to extract all of the parameters of interest about x[n] from the eigenvalues and eigenvectors of $\mathbf{R}_X$ as follows:

1. perform an eigendecomposition of the autocorrelation matrix $\mathbf{R}_X$, the largest eigenvalue is equal to $MP_1+\sigma_w^2$ and the remaining eigenvalues are equal to σ_w^2
2. use the eigenvalues of $\mathbf{R}_X$ to solve for the power P_1 and noise variance:

$$\boxed{\sigma_W^2 = \lambda_{\min} \qquad , \qquad P_1 = \frac{\lambda_{\max} - \lambda_{\min}}{M}}$$

3. determine the frequency ω_1 from the eigenvector $\mathbf{v}_{max}$ that is associated with the largest eigenvalue using, for example, the second coefficient of $\mathbf{v}_{max}$ (recall the definition of $\mathbf{e}_1$ in the previous slide), or using a more robust approach (see next)

Since the eigenvalues and eigenvectors may be quite sensitive to small errors in $r_X[k]$, instead of estimating the frequency of the complex exponential from a single eigenvector, an approach based on a *pseudo-spectrum* may be more appropriate (see next).

spectrum estimation

If $\mathbf{v}_i$ is a noise eigenvector of $\mathbf{R}_X$ and therefore its eigenvalue is σ_w^2, then the orthogonality property $\boxed{\mathbf{e}_1^H \mathbf{v}_k = 0, \quad k = 2,3,\ldots,M}$ means that the Fourier

transform $\boxed{V_i\left(e^{j\omega}\right) = \sum_{\ell=0}^{M-1} v_i\, e^{-j\ell\omega} = \mathbf{e}^H \mathbf{v}_i}$ must have a zero at $\omega=\omega_1$, which is the

frequency of the sinusoid we want to estimate. Therefore, if we form the frequency estimation function (a pseudo-spectrum):

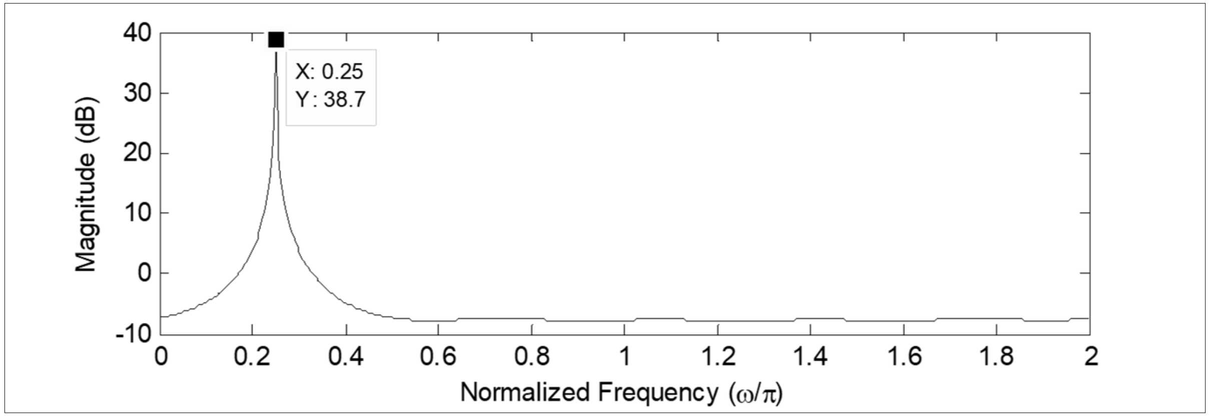
$$\boxed{\hat{P}_i\left(e^{j\omega}\right) = \frac{1}{\left|\sum_{k=0}^{M-1} v_i[k]\, e^{-jk\omega}\right|^2} = \frac{1}{\left|\mathbf{e}^H \mathbf{v}_i\right|^2}}$$

then $P_i(e^{j\omega})$ will be very large at $\omega=\omega_1$, and the location of the peak may be used to estimate the frequency of the complex exponential. In order to avoid the sensitivity to errors in the estimation of $\mathbf{R}_X$, a weighted average of all of the noise eigenvectors delivers a more robust estimate:

$$\boxed{\hat{P}\left(e^{j\omega}\right) = \frac{1}{\sum_{i=2}^{M} \alpha_i \left|\mathbf{e}^H \mathbf{v}_i\right|^2}}$$

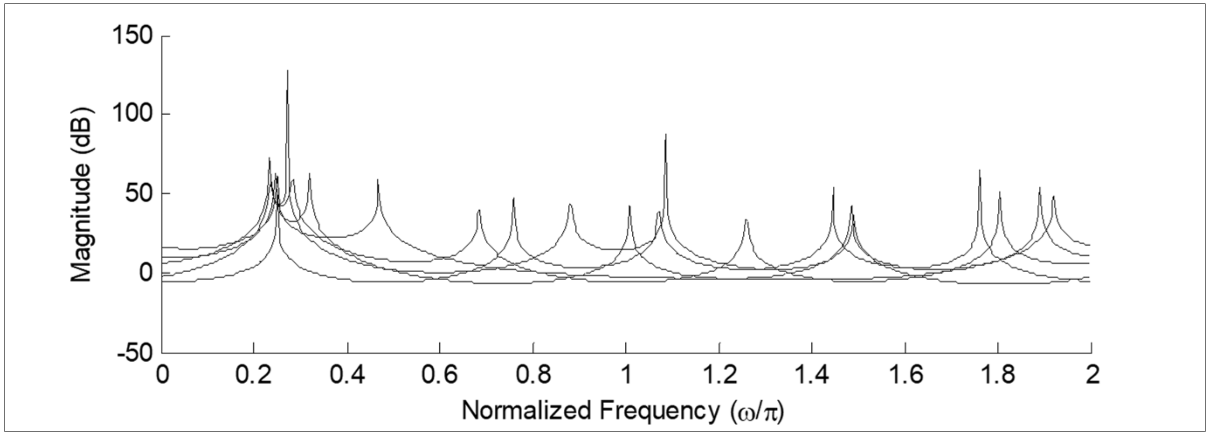
spectrum estimation

As an example, if $x[n] = 4e^{-j(n\pi/4+\phi)} + w[n]$, taking N=64 values a 6×6 autocorrelation matrix is estimated and an eigendecomposition is performed. Using α_i=1, the average (pseudo) spectrum (evaluated using 512 points) is



and the peak clearly indicates the correct normalized frequency: ¼.

For comparison, an overlay plot of the individual five pseudo spectra is



Note: the Matlab code implementing this example is in the next slide

spectrum estimation

The previous figures have been generated using the following Matlab code

```
N=64; n=[0:N-1]; M=6;
phi=2*(rand-0.5)*pi; omega=pi/4;
x=4*exp(j*(n*omega+phi))+randn(1,N);
R=covar(x,M);
[V,D]=eig(R);
for k=1:M, diagonal(k)=D(k,k); end
[sorteddiag indx]=sort(diagonal,'descend');
% indx(1) has the largest
xfreq=[0:511]*2/512;
figure(1); accffft=zeros(512,1);
for k=2:M
    hold on
    plot(xfreq,20*log10(1./abs(fft(V(:,indx(k)),512))));
    hold off
    accffft=accffft+abs(fft(V(:,indx(k)),512)).^2;
end
xlabel('Normalized Frequency (\omega/\pi)')
ylabel('Magnitude (dB)');

figure(2)
plot(xfreq,10*log10(1./accffft))
xlabel('Normalized Frequency (\omega/\pi)')
ylabel('Magnitude (dB)');
```

spectrum estimation

As a generalization, we consider the case of a wide-sense stationary process consisting of *p* distinct complex exponentials in white noise. The M×M autocorrelation sequence is

$$r_X[k] = \sum_{\ell=1}^{p} P_\ell \, e^{jk\omega_\ell} + \sigma_W^2 \delta[\ell]$$

where $P_\ell = |A_\ell|^2$. The autocorrelation matrix can be written as

$$\mathbf{R}_X = \mathbf{R}_{SIG} + \mathbf{R}_{NOI} = \sum_{i=1}^{p} P_i \mathbf{e}_i \mathbf{e}_i^H + \sigma_W^2 \mathbf{I}$$

where $\mathbf{e}_i = \left[1, \quad e^{j\omega_i}, \quad e^{j2\omega_i}, \ldots \quad , e^{j(M-1)\omega_i}\right]^T \qquad , \qquad i = 1,2,\ldots,p$

consists of a set of *p* linearly independent vectors. The above equation may be written as $\mathbf{R}_X = \mathbf{EPE}^H + \sigma_W^2 \mathbf{I}$ where $\mathbf{E} = \left[\mathbf{e}_1, \ldots \quad , \mathbf{e}_p\right]$ is an M×p matrix containing the *p* signal vectors and **P**=diag{P_1, P_2, …, P_p}.

Since the eigenvalues of $\mathbf{R}_X$ are $\lambda_i^{SIG} + \sigma_W^2$ where λ_i^{SIG} are the eigenvalues of $\mathbf{R}_{SIG}$, and since matrix $\mathbf{R}_{SIG}$ is a matrix of rank *p*, then, the first group of p eigenvalues of $\mathbf{R}_x$ are greater than σ_w^2 and the second group with the last M-p eigenvalues are equal (in theory) to σ_w^2.

spectrum estimation

The eigenvectors of the first group are the signal eigenvectors and the eigenvectors of the second group are the noise eigenvectors. Assuming eigenvectors are normalized to have unit norm, then the spectral theorem may be used to decompose $\mathbf{R}_X$

$$\mathbf{R}_X = \sum_{i=1}^{p} \left(\lambda_i^{SIG} + \sigma_W^2 \right) \mathbf{v}_i \mathbf{v}_i^H + \sum_{i=p+1}^{M} \sigma_W^2 \mathbf{v}_i \mathbf{v}_i^H$$

The orthogonality of the signal and noise subspaces may be used to estimate the frequencies of the complex exponentials. Each signal vector $\mathbf{e}_1$, …, $\mathbf{e}_p$, is orthogonal to each of the noise eigenvectors:

$$\mathbf{e}_i^H \mathbf{v}_k = 0 \quad , \quad i = 1, 2, \ldots p \quad , \quad k = p+1, p+2, \ldots, M$$

Therefore, the frequencies may be estimated using a frequency estimation function such as

$$\hat{P}\left(e^{j\omega}\right) = \frac{1}{\sum_{i=p+1}^{M} \alpha_i \left| \mathbf{e}^H \mathbf{v}_i \right|^2}$$

- *Pisarenko Harmonic Decomposition*
  - In 1973 Pisarenko demonstrated that the frequencies of complex exponentials in white noise could be derived from the eigenvector corresponding to the minimum eigenvalue of the autocorrelation matrix
  - In the Pisarenko harmonic decomposition, x[n] is a sum of *p* complex exponentials in white noise and the number of complex exponentials is known, it is also assumed that *p+1* values of the autocorrelation sequence are either known or have been estimated
  - the $(p+1)\times(p+1)$ autocorrelation matrix implies that the dimension of the noise subspace is equal to one, and is spanned by the eigenvector corresponding to the minimum eigenvalue, $\lambda_{min} = \sigma_w^2$, if the corresponding eigenvector is $\mathbf{v}_{min}$, then $\mathbf{v}_{min}$ will be orthogonal to each one of the signal eigenvectors $\mathbf{e}_\ell$

$$\mathbf{e}_\ell^H \mathbf{v}_{\min} = \sum_{k=0}^{p} v_{\min}[k] e^{-jk\omega_\ell} = 0 \quad , \quad \ell = 1,2,\ldots,p$$

thus, $V_{min}(e^{j\omega})$ is equal to zero at each of the harmonic complex exponential frequencies $\omega=\omega_\ell$ for $\ell=1,2,\ldots,p$

$$V_{\min}\left(e^{j\omega}\right) = \sum_{k=0}^{p} v_{\min}[k] e^{-jk\omega}$$

spectrum estimation

This result means that the Z transform of the noise eigenvector (or eigenfilter), has *p* zeros on the unit circle

$$V_{\min}(z) = \sum_{k=0}^{p} v_{\min}[k] z^{-k} = \prod_{\ell=1}^{p} \left(1 - e^{j\omega_\ell} z^{-\ell}\right)$$

and, therefore, the frequencies of the complex exponentials may be extracted from the roots of the eigenfilter; as an alternative (as and seen before), the frequency estimation function, which is also known as *eigenspectrum* or *pseudospectrum* (because it has the form of a power spectrum but it does not contain any information about the power in the complex exponentials):

$$\hat{P}_{PHD}\left(e^{j\omega}\right) = \frac{1}{\left|\mathbf{e}^H \mathbf{v}_{\min}\right|^2}$$

will be large (in theory, infinite) at the frequencies of the complex exponentials, therefore the peaks of $P_{PHD}(e^{j\omega})$ may be used to estimate the frequencies of the complex exponentials

spectrum estimation

Once the frequencies of the complex exponentials have been found, the powers P_i may be determined from the eigenvalues of R_X. Assuming that the signal subspace vectors $v_1, v_2, \ldots, v_p$ have been normalized so that $\mathbf{v}_i^H\mathbf{v}_i=1$, then

$$\mathbf{R}_X \mathbf{v}_i = \lambda_i \mathbf{v}_i \quad ; \quad i=1,2,\ldots,p \qquad \Leftrightarrow \qquad \mathbf{v}_i^H \mathbf{R}_X \mathbf{v}_i = \lambda_i \mathbf{v}_i^H \mathbf{v}_i = \lambda_i$$

using (slide 53) $\mathbf{R}_X = \mathbf{R}_{SIG} + \mathbf{R}_{NOI} = \sum_{k=1}^{p} P_k \mathbf{e}_k \mathbf{e}_k^H + \sigma_W^2 \mathbf{I}$, then

$$\mathbf{v}_i^H \mathbf{R}_X \mathbf{v}_i = \mathbf{v}_i^H \left(\sum_{k=1}^{p} P_k \mathbf{e}_k \mathbf{e}_k^H + \sigma_W^2 \mathbf{I} \right) \mathbf{v}_i = \left(\sum_{k=1}^{p} P_k (\mathbf{e}_k^H \mathbf{v}_i)^H \mathbf{e}_k^H \mathbf{v}_i + \sigma_W^2 \mathbf{v}_i^H \mathbf{v}_i \right) = \sum_{k=1}^{p} P_k \left| \mathbf{e}_k^H \mathbf{v}_i \right|^2 + \sigma_W^2 = \lambda_i$$

and, therefore $\sum_{k=1}^{p} P_k \left| \mathbf{e}_k^H \mathbf{v}_i \right|^2 = \lambda_i - \sigma_W^2 \quad ; \quad i=1,2,\ldots,p$; in addition, recognizing

that $\left| \mathbf{e}_k^H \mathbf{v}_i \right|^2 = \left| \sum_{\ell=0}^{p} v_i[\ell] e^{-j\ell\omega_k} \right|^2 = \left| V_i\left(e^{j\omega_k}\right) \right|^2 \quad , \quad i=1,2,\ldots,p$, the previous equation

may be written as $\sum_{k=1}^{p} P_k \left| V_i\left(e^{j\omega_k}\right) \right|^2 = \lambda_i - \sigma_W^2 \quad ; \quad i=1,2,\ldots,p$

Or, in a matrix form:

$$\begin{bmatrix} \left|V_1\left(e^{j\omega_1}\right)\right|^2 & \left|V_1\left(e^{j\omega_2}\right)\right|^2 & \cdots & \left|V_1\left(e^{j\omega_p}\right)\right|^2 \\ \left|V_2\left(e^{j\omega_1}\right)\right|^2 & \left|V_2\left(e^{j\omega_2}\right)\right|^2 & \cdots & \left|V_2\left(e^{j\omega_p}\right)\right|^2 \\ \vdots & \vdots & & \vdots \\ \left|V_p\left(e^{j\omega_1}\right)\right|^2 & \left|V_p\left(e^{j\omega_2}\right)\right|^2 & \cdots & \left|V_p\left(e^{j\omega_p}\right)\right|^2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_p \end{bmatrix} = \begin{bmatrix} \lambda_1 - \sigma_W^2 \\ \lambda_2 - \sigma_W^2 \\ \vdots \\ \lambda_p - \sigma_W^2 \end{bmatrix}$$

which denotes a set of *p* linear equations in the *p* unknowns, P_k (which may be solved for the parameters P_k)

Note: Munson Hayes provides (page 461) a Matlab function `phd()` allowing to estimate the frequencies of *p* complex exponentials in white noise

Despite its elegance, the Pisarenko harmonic decomposition is not commonly used in practice because

- it requires that the number of complex exponentials is exactly known,
- the frequency estimates are biased in case the additive noise is not white

spectrum estimation

- *MULtiple SIgnal Classification method (MUSIC)*
  - The MUSIC algorithm algorithm is a frequency estimation technique that is a generalization of the Pisarenko algorithm

Assuming that x[n] is a random process consisting of *p* complex exponentials in white noise having variance σ_w^2, an M×M correlation matrix $\mathbf{R}_X$ with M>(p+1), eigenvalues arranged in decreasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M$ and corresponding eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M$;

the eigenvectors may be separated into two groups: the *p* eigenvectors corresponding to the *p* largest eigenvalues (signal subspace), and the M-p remaining eigenvectors (noise subspace) corresponding to eigenvalues that, in practice (due to inexact autocorrelations), are approximately equal to σ_w^2, an estimate of σ_w^2 may be obtained by averaging the M-p smallest eigenvalues

$$\boxed{\hat{\sigma}_W^2 = \frac{1}{M-p}\sum_{k=p+1}^{M}\lambda_k}$$

spectrum estimation

The eigenvectors of $\mathbf{R}_X$ are of length M, which means each noise eigenfilter has M-1 roots:

$$V_i(z) = \sum_{k=0}^{M-1} v_i[k] z^{-k} \qquad , \qquad i = p+1, \ldots, M$$

p roots will lie on the unit circle at the frequencies of the complex exponentials, and the eigenspectrum associated with the noise eigenvector $\mathbf{v}_i$ will exhibit sharp peaks at the frequencies of the complex exponentials

$$\frac{1}{\left|V_i\left(e^{j\omega}\right)\right|^2} = \frac{1}{\left|\mathbf{e}^H \mathbf{v}_i\right|^2} = \frac{1}{\left|\sum_{k=0}^{M-1} v_i[k] e^{-jk\omega}\right|^2}$$

there are two sources of inaccuracies:

1. the remaining M-p-1 zeros may lie anywhere in the z plane and, if close to the unit circle, they may give rise to spurious peaks, in the eigenspectrum,
2. due to inexact autocorrelations, the zeros of $V_i(z)$ that should be on the unit circle, may not remain there

Therefore, when only one noise eigenvector is used to estimate the complex exponential frequencies, there may be some ambiguity in distinguishing the desired peaks from the spurious ones

spectrum estimation

Since the spurious peaks that are introduced from each of the noise subspace filters tend to occur at different frequencies (as already illustrated before (slide 52)), their effect is reduced in the MUSIC algorithm by averaging:

$$\hat{P}_{MU}\left(e^{j\omega}\right)=\frac{1}{\sum_{i=p+1}^{M}\left|\mathbf{e}^{H}\mathbf{v}_{i}\right|^{2}}$$

The frequencies of the complex exponentials are taken as the locations of the p largests peaks in $V_{MU}(e^{j\omega})$; alternatively the root MUSIC method may be used: since the z-transform equivalent of $P_{MU}(e^{j\omega})$ is $P_{MU}(z)$:

$$\hat{P}_{MU}\left(z\right)=\frac{1}{\sum_{i=p+1}^{M}V_{i}(z)V_{i}^{*}(1/z^{*})}$$

then, the frequency estimates may be taken as the angles of the *p* roots of the denominator polynomial of $P_{MU}(z)$; in either case, the power of each complex exponential may then be found using the matrix as in slide 58

Note: Munson Hayes provides (page 465) a Matlab function `music()` allowing to estimate the frequencies of *p* complex exponentials in white noise

spectrum estimation

- *other eigenvector methods*

other methods have been proposed for estimating the frequencies of complex exponentials in noise, e.g. the EigenVector method (EV) uses

$$\hat{P}_{EV}\left(e^{j\omega}\right) = \frac{1}{\sum_{i=p+1}^{M} \frac{1}{\lambda_i} \left| \mathbf{e}^H \mathbf{v}_i \right|^2}$$

where λ_i is the eigenvalue associated with the eigenvector $\mathbf{v}_i$; another eigendecomposition-based method is the *minimum norm algorithm* that uses a single vector **a** that is constrained to lie in the noise subspace and is subject to the following contraints

1. vector **a** lies in the noise subspace (ensures that p roots of A(z) lie on the unit circle)
2. vector **a** has minimum norm (ensures that the spurious roots of H(z) lie inside the unit circle)
3. the first element of **a** is unity (ensures that the zero vector is not a solution)

$$\hat{P}_{MN}\left(e^{j\omega}\right) = \frac{1}{\left| \mathbf{e}^H \mathbf{a} \right|^2}$$

- *Principal Components Spectrum Estimation*

In previous methods, referred to as *noise subspace methods* (because only the vectors that lie in the noise subspace are used), the orthogonality between signal and noise subspaces was exploited to estimate the frequencies of *p* complex exponentials in noise

Another class of methods exits, referred to as *signal subspace methods*, that are base on a principal components analysis of the autocorrelation matrix

As in the MUSIC method, we assume that x[n] is a random process consisting of *p* complex exponentials in white noise having variance σ_w^2, and that $\mathbf{R}_X$ is an M×M correlation matrix with M>(p+1), whose eigenvalues are arranged in decreasing order $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M$ (and the corresponding eigenvectors are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_M$)

The eigendecomposition of $\mathbf{R}_X$ leads to

$$\mathbf{R}_X = \sum_{k=1}^{M} \lambda_k \mathbf{v}_k \mathbf{v}_k^H = \sum_{k=1}^{p} \lambda_k \mathbf{v}_k \mathbf{v}_k^H + \sum_{k=p+1}^{M} \lambda_k \mathbf{v}_k \mathbf{v}_k^H$$

spectrum estimation

By retaining only the principal components of $\mathbf{R}_X$, i.e. the principal eigenvectors of $\mathbf{R}_X$, a reduced rank approximation to $\mathbf{R}_X$ is formed:

$$\hat{\mathbf{R}}_X = \sum_{k=1}^{p} \lambda_k \mathbf{v}_k \mathbf{v}_k^H$$

this matrix has rank *p* and and may be used in a principal components-based method such as the Blackman-Tukey, minimum variance method, or maximum entropy method, in order to estimate the power spectrum

Blackman-Tukey frequency estimation

The power spectrum is estimated as $\hat{P}_{BT}\left(e^{j\omega}\right) = \sum_{k=-M}^{M} \hat{r}_X[k] w[k] e^{-jk\omega}$; if w[n] is the rectangular window, then the blackman-Tukey estimate may be written as

$$\hat{P}_{BT}\left(e^{j\omega}\right) = \frac{1}{M} \sum_{k=-M}^{M} \left(1-|k|\right)\hat{r}_X[k] e^{-jk\omega} = \frac{1}{M} \mathbf{e}^H \mathbf{R}_X \mathbf{e} \qquad (?)$$

an eigendecomposition of the autocorrelation matrix leads to

$$P_{BT}\left(e^{j\omega}\right) = \frac{1}{M} \sum_{k=1}^{M} \lambda_k \left|\mathbf{e}^H \mathbf{v}_k\right|^2$$

spectrum estimation

A principal components version of this power spectrum estimate is

$$\hat{P}_{PC-BT}\left(e^{j\omega}\right)=\frac{1}{M}\mathbf{e}^H\hat{\mathbf{R}}_X\mathbf{e}=\frac{1}{M}\sum_{k=1}^{p}\lambda_k\left|\mathbf{e}^H\mathbf{v}_k\right|^2$$

Minimum variance frequency estimation

As indicated in slide 33, if $r_X[k]$ is an autocorrelation sequence of a random process for lags $|k|\leq M$, the Mth-order minimum variance spectrum estimate is

$$\hat{P}_{MV}\left(e^{j\omega}\right)=\frac{M}{\mathbf{e}^H\mathbf{R}_X^{-1}\mathbf{e}}$$

The eigendecomposition of $\mathbf{R}_X$ leads to the inverse $\mathbf{R}_X^{-1}$

$$\mathbf{R}_X^{-1}=\sum_{k=1}^{M}\frac{1}{\lambda_k}\mathbf{v}_k\mathbf{v}_k^H=\sum_{k=1}^{p}\frac{1}{\lambda_k}\mathbf{v}_k\mathbf{v}_k^H+\sum_{k=p+1}^{M}\frac{1}{\lambda_k}\mathbf{v}_k\mathbf{v}_k^H$$

As a result, retaining only the first *p* principal components of $\mathbf{R}_X^{-1}$ leads to the principal components minimum variance estimate

$$\hat{P}_{PC-MV}\left(e^{j\omega}\right)=\frac{1}{\sum_{k=1}^{p}\frac{1}{\lambda_k}\left|\mathbf{e}^H\mathbf{v}_k\right|^2}$$

Note: this provides an estimate of the power spectrum whereas a similar result related to the EigenVector method (slide 63) only provides a pseudo-spectrum allowing to estimate the frequencies of complex exponentials

Autoregressive frequency estimation

Autoregressive spectrum estimation using the autocorrelation, covariance, or modified covariance algorithms, involves finding the solution to a set of linear equations of the form $\mathbf{R}_X\mathbf{a}_M = \varepsilon_M \mathbf{u}_1$; R_X is an $(M+1)\times(M+1)$ autocorrelation matrix, hence $\mathbf{a}_M = \varepsilon_M \mathbf{R}_X^{-1}\mathbf{u}_1$ which leads to the power spectrum estimate

$$\hat{P}_{AR}\left(e^{j\omega}\right) = \frac{\left|b[0]\right|^2}{\left|\mathbf{e}^H \mathbf{a}_M\right|^2}$$

where $|b[0]|^2=\varepsilon_M$, however since x[n] consists of *p* complex sinusoids, then a principal components solution for a_M is

$$\mathbf{a}_M = \varepsilon_M \mathbf{R}_X^{-1}\mathbf{u}_1 = \varepsilon_M \left(\sum_{k=1}^{p} \frac{1}{\lambda_k} \mathbf{v}_k \mathbf{v}_k^H\right)\mathbf{u}_1 = \varepsilon_M \sum_{k=1}^{p} \frac{v_k^*[0]}{\lambda_k} \mathbf{v}_k = \varepsilon_M \sum_{k=1}^{p} \alpha_k \mathbf{v}_k$$

which leads to

$$\hat{P}_{PC-AR}\left(e^{j\omega}\right) = \frac{1}{\left|\sum_{k=1}^{p} \alpha_k \mathbf{e}^H \mathbf{v}_k\right|^2}$$

Note: even if the order of the autocorrelation vector increases, only p principal eigenvectors and eigenvalues are used in the spectrum estimate, which avoids the increase spurious peaks due the noise eigenvectors of the autocorrelation matrix

spectrum estimation

Summary and word of caution

Non-parametric methods

- the periodogram is not a consistent power spectrum estimator
- periodogram averaging or smoothing provides a consistent estimate of the power spectrum but has inherent difficulties dealing with short data records
- have the advantage that they can be used with any type of process
- periodogram-based frequency estimation is possible by taking advantage of the knowledge of the data window
- minimum variance and maximum entropy methods seek to improve the resolution and “whiteness” of periodogram-based methods, respectively

Parametric methods

- presume a priori information concerning the model that explains the data
- after the parameters have been found, they can be incorporated into the parametric form for the spectrum
- unless the model that is used is consistent with the data that is analysed, inaccurate or misleading spectrum estimates may result

Frequency and spectrum estimation using eigendecomposition

- frequency (and possibly power) estimation using the pseudospectrum takes advantage of the fact that signal and noise subspaces are orthogonal
- frequency and spectrum estimation using principal components analysis use a reduced rank approximation to the autocorrelation matrix