L.EEC/M.EEC

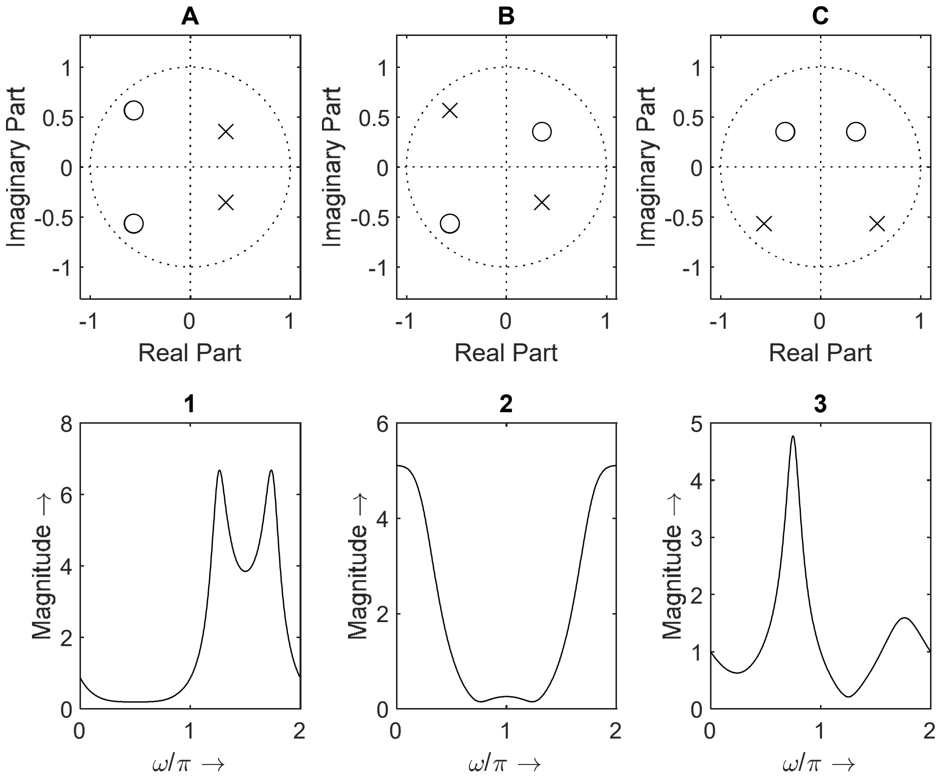
L.EEC025 - Fundamentals of Signal Processing

SECOND EXAM, JANUARY 28, 2025

Duration: 120 Minutes, closed book

NOTE: each question ***must*** be answered in a separate sheet; please write your name and Student order number on all sheets, please provide complete answers while trying to minimize paper usage. Except for a basic scientific calculator and the provided formulae sheet, no other materials or tools, including (so-called) AI assistants, are allowed in this exam.
DISCLAIMER: no FunSP materials, including this exam, have been produced using any (so-called) AI-based tools

1. Three different causal discrete-time systems have zero-pole diagrams A, B, and C, and the frequency response magnitudes 1, 2, and 3, as represented next. The radius of each represented pole, or zero, is either 0.5 or 0.8, and the angle of each represented pole, or zero, is a multiple of $\pi/4$ rad.



a) **[1,5 *pts*]** Match each zero-pole diagram (A, B, C) to the corresponding frequency response magnitude (1, 2, 3), and indicate the main supporting arguments.

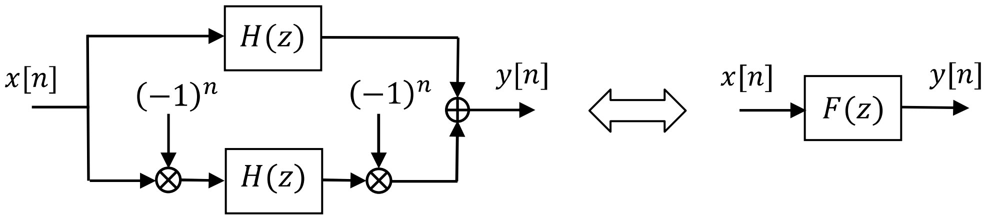
b) **[0.5 *pts*]** Which of the represented systems (A, B, or C) are minimum-phase ? And which ones have a real-valued impulse response ? Why ?

c) **[1 *pt*]** Admit that $H_A(z)$ and $H_B(z)$ represent the transfer functions of systems A and B, respectively, and that a new system is obtained as $F(z) = H_A(z)H_B(z)$. What is the order of $F(z)$? Make a plausible sketch of its frequency response magnitude.

d) **[1,5 *pts*]** The impulse responses of two of the above systems are represented by $h_X[n]$ and $h_Y[n]$, where $X, Y \in \{A, B, C\}$. If G represents a constant gain, is it true, for some constant α, that $(\alpha)^n h_X[n] * h_Y[n] = G\delta[n]$? If yes, identify X and Y, and find α.

Hint: Develop your reasoning using an analysis in the Z-domain.

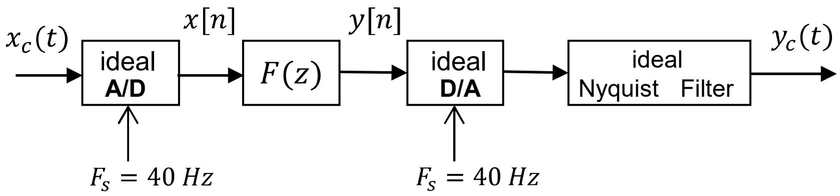
2. Consider the illustrated system equivalence.



a) **[1,5 *pts*]** Show that $F(z) = H(z) + H(-z)$.

b) **[1,5 *pt*]** Admit that $H(z) = \frac{1+z^{-1}}{1-\alpha z^{-1}}$, $|z| > \alpha$, where α is a real positive constant, $\alpha < 1$. Show that $F(z) = 2\frac{1+\alpha z^{-2}}{1-\alpha^2 z^{-2}}$, and find the frequencies for which $|F(e^{j\omega})|$ is maximum, and for which it is minimum.

c) **[1 *pt*]** Consider the illustrated analog and causal discrete-time system whose transfer function is $F(z)$, as suggested in **a)**. An *anti-aliasing* filter does not exist and the input analog signal is $x_c(t) = 1 + \sin(60\pi t) - \sin(100\pi t)$.



Find the sinusoidal frequencies of the discrete-time signal $x[n]$ in the Nyquist range, i.e., in the range $-\pi \le \omega < \pi$. Obtain a compact expression for $x[n]$.

d) **[1 *pt*]** Presuming ideal reconstruction conditions, indicate what sinusoidal frequencies (in Hertz) exist in $y_c(t)$, and indicate what their magnitudes are.

Note: In case you did not solve **c)**, admit that the $x[n]$ sinusoidal frequencies are $\omega_0 = 0$ rad., $\omega_1 = \frac{\pi}{2}$ rad.

3. Consider the following Matlab code.

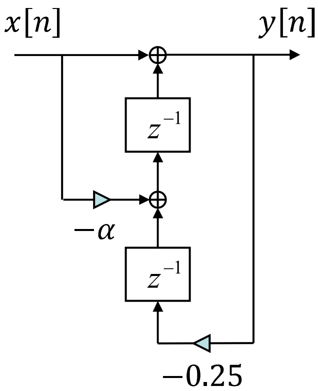
```
x=[1  2  3  4j  5j  6j];
X=fft(x); N=length(X); Y=zeros(size(X));
Y(1)=conj(X(1)); Y(2:N)=conj(X(N:-1:2));
A=(X+Y)/2; B=(X-Y)/2; a=ifft(A)
C=A.*B; ifft(C)
D=zeros(size(A));
D(1)=A(1); D(2:N)=A(N:-1:2);
F=(A+D)/2; ifft(F)
```

a) **[1 *pt*]** Express $a[n]$ as a function of $x[n]$. Find the result of `ifft(A)` without executing the code.

b) **[1,5 *pts*]** Express $c[n]$ as a function of $x[n]$. Find the result of `ifft(C)` without executing the code.

c) **[1 *pt*]** Express $f[n]$ as a function of $a[n]$. Explain. (**Note**: you don't need to compute $f[n]$.)

4. The realization structure of a causal discrete-time system is depicted next where α is a constant, $|\alpha| < 1$.

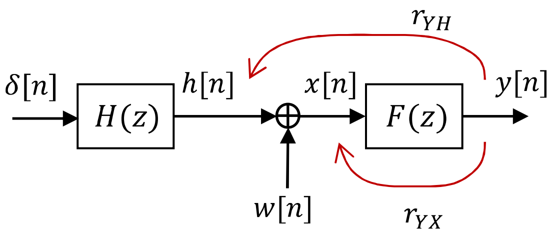


a) **[1 *pt*]** Obtain a difference equation characterizing the system and obtain its transfer function (including the RoC).

b) **[1,5 *pts*]** Determine the output sequence when the input sequence exciting the system is $x[n] = \alpha^n u[n]$.

c) **[1 *pt*]** Admit that each unitary delay (z^{-1}) in the delay chain of the realization structure of the system is replaced by $-z^{-1}$. What is the impact of this modification to the order of the system, and to its frequency response magnitude ? Admitting that modification, what input sequence would generate the same output as in **b)** ?

5. Consider the illustrated block diagram where $H(z)$ represents the transfer function of a gain-normalized first-order all-pass system, $H(z) = \frac{z^{-1}-\alpha^*}{1-\alpha z^{-1}}$, $|z| > |\alpha|$, $|\alpha| < 1$, and $w[n]$ represents white noise whose autocorrelation is the unit impulse, i.e., $r_W[\ell] = \delta[\ell]$.



a) **[1 *pt*]** Find the impulse response of the all-pass system, $h[n]$, and show that its autocorrelation is $r_H[\ell] = h_H[\ell] * h_H^*[-\ell] = \delta[\ell]$.

Hint: Regarding $r_H[\ell]$, develop your reasoning in the Z-domain.

b) **[1,5 *pts*]** Show that if two uncorrelated signals are added: $x[n] = h[n] + w[n]$, with $h[\ell] * w^*[-\ell] = w[\ell] * h^*[-\ell] = 0$, then the autocorrelation of $x[n]$ is given by the sum of the individual autocorrelations, i.e., $r_X[\ell] = r_H[\ell] + r_W[\ell]$. Under this assumption, and using the above information, find $r_X[\ell]$.

c) **[1 *pt*]** It is known that if $F(z)$ represents a linear and shift-invariant system, then the cross-correlation between its output sequence, $y[\ell]$, and its input sequence, $x[\ell]$, is given by $r_{YX}[\ell] = f[\ell] * r_X[\ell]$, where $f[\ell]$ is the impulse response of the system. If $r_{YX}[\ell] = 6\delta[\ell-2] - 4\delta[\ell-3] + 2\delta[\ell-4]$, and using the above information, find $f[\ell]$. Would it be possible to find $f[\ell]$ by computing $r_{YH}[\ell]$?

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