1. a) B – 3 is the easiest association to identify because the two pole-zero all-pass pairs determine a flat frequency response magnitude, a zero being the reciprocal-conjugate of its pole pair.

A – 2
C – 1
the difference between these two cases is that in A the poles are closer to the zeros than in C; the impact of this is that the effect of the zeros is de-emphasized for frequencies other those the zeros are aligned with ($\pm\pi/3$ rad), which creates a notch filter behavior; thus, in A-2 the gains in the frequency response magnitude for $\omega = 0$ rad and for $\omega = \pi$ rad are closer to each other than they are in C-1.

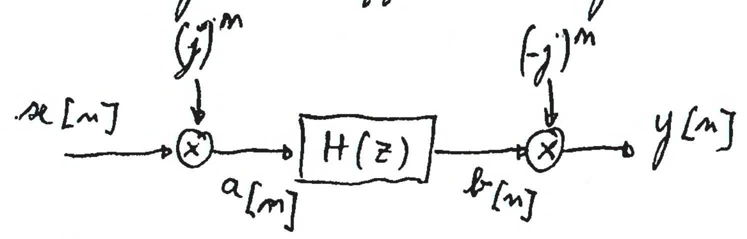
b) All systems are stable because the poles are all inside the unit circumference; all systems have a real-valued impulse response as all poles and zeros exist as complex-conjugate pairs.

c) Systems A and C belong to a common category of systems named "notch filters", their purpose is to reject a desired (and specific) frequency, which is achieved by placing a zero on the unit circumference according to that frequency (i.e., angle). Thus, they can be used to eliminate a sinusoidal interference having a well-defined frequency.

System B is an all-pass system, as such, its main characteristic is the group delay (as the frequency response magnitude is flat) which is the same as writing: the non-linear phase response. Thus, it can be used as a compensator of a desired group delay distortion, for example, aiming at an overall flat group-delay response (which is the same as writing: linear-phase response).

d) Given the indicated possibilities for the radii of poles and zeros, it is clear that the radii of the poles in system A is 0.25 while the radii of the poles in system B is 0.5. On the other hand, the radii of the zeros in system A is 1 while the zeros in system B is 2. Also, the orientation of poles and zeros in both systems is the same: $+\pi/3$ rad. or $-\pi/3$ rad. This means that the poles and zeros of system B are just scaled versions of the poles and zeros of system A: $Z_B = 2 Z_A$. Thus: $H_B(z) = H_A\left(\frac{z}{2}\right)$ which, according to the properties of the Z-Transform, means that: $h_B[n] = 2^n h_A[n]$.

2. a) Considering the different signals involved:



we have :

$$a[n] = (j)^n x[n] \xleftrightarrow{Z} A(z) = X\left(\frac{z}{j}\right)$$

$$b[n] = h[n] * a[n] \longleftrightarrow B(z) = H(z)A(z)$$

$$= H(z)X\left(\frac{z}{j}\right)$$

$$y[n] = (-j)^n b[n] \longleftrightarrow Y(z) = B\left(\frac{z}{-j}\right)$$

$$= H\left(\frac{z}{-j}\right)X\left(\frac{z}{j(-j)}\right)$$

$$= H(jz)X(z)$$

Thus, $F(z) = H(jz)$

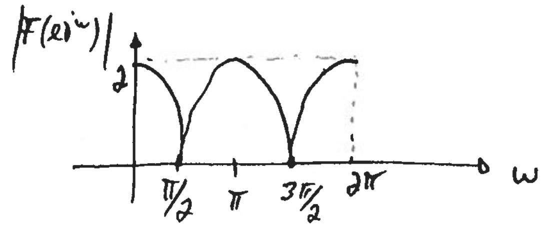
b) $H(z) = (1-z^{-1})(1+z^{-1})$ and, as $F(z) = H(jz)$, we have:

$$F(z) = \left(1-(jz)^{-1}\right)\left(1+(jz)^{-1}\right) = (1+jz^{-1})(1-jz^{-1}) = 1+z^{-2}$$

from which we obtain :

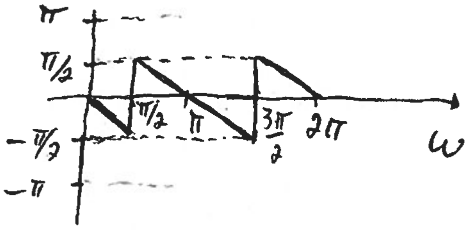
$$F(e^{j\omega}) = 1+e^{-j2\omega} = e^{-j\omega}\left(e^{j\omega}+e^{-j\omega}\right) = 2e^{-j\omega}\cos\omega$$

$$= |F(e^{j\omega})|\, e^{j\angle F(e^{j\omega})}$$



where $|F(e^{j\omega})| = 2|\cos\omega|$

and $\angle F(e^{j\omega}) = -\omega +$ jumps of $\pm\pi$



c) $x_c(t) = 1 + \sin 490\pi t + \cos 420\pi t$

$$x[n] = x_c(t)\Big|_{t = nT_s = \frac{n}{F_s}} = 1 + \sin 490\pi \frac{n}{280} + \cos 420\pi \frac{n}{280}$$

$$= 1 + \sin n\frac{7\pi}{4} + \cos n\frac{6\pi}{4} = e^{jn\omega_0} + \sin n\omega_1 + \cos n\omega_2$$

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which means that:

$\omega_0 = 0$ rad.

$\omega_1 = \frac{7\pi}{4}$ rad. , as $|\omega_1| > \pi$ $\therefore$ $\omega_1 = \frac{7\pi}{4} + K2\pi = \left.\frac{7\pi + K8\pi}{4}\right|_{K=-1} = -\frac{\pi}{4}$ rad.

$\omega_2 = \frac{6\pi}{4}$ rad. , as $|\omega_2| > \pi$ $\therefore$ $\omega_2 = \frac{6\pi}{4} + K2\pi = \left.\frac{6\pi + K8\pi}{4}\right|_{K=-1} = -\frac{\pi}{2}$ rad.

which are the final frequencies within the Nyquist range, as a result: $x[n] = 1 + \sin\left(-n\frac{\pi}{4}\right) + \cos\left(-n\frac{\pi}{2}\right)$

$$= 1 - \sin n\frac{\pi}{4} + \cos n\frac{\pi}{2}.$$

c) As the input signal components are all real-valued sinusoidal components,

$$y[n] = |F(e^{j0})|\, e^{j(n\omega_0 + \angle F(e^{j0}))} - |F(e^{j\frac{\pi}{4}})| \sin\left(n\frac{\pi}{4} + \angle F(e^{j\frac{\pi}{4}})\right)$$
$$+ |F(e^{j\frac{\pi}{2}})| \cos\left(n\frac{\pi}{2} + \angle F(e^{j\frac{\pi}{2}})\right)$$

and considering that

$F(e^{j0}) = 2$

$F(e^{j\frac{\pi}{4}}) = 2\, e^{-j\frac{\pi}{4}} \times \frac{\sqrt{2}}{2} = \sqrt{2}\, e^{-j\frac{\pi}{4}}$ $\therefore$ $|F(e^{j\frac{\pi}{4}})| = \sqrt{2}$, $\angle F(e^{j\frac{\pi}{4}}) = -\frac{\pi}{4}$

$F(e^{j\frac{\pi}{2}}) = 2\, e^{-j\frac{\pi}{2}} \times 0 = 0$

it results that $y[n] = 2 - \sqrt{2} \sin\left(n\frac{\pi}{4} - \frac{\pi}{4}\right)$

and, presuming ideal reconstruction conditions,

$$y[n] = y_c(t)\Big|_{t=\frac{n}{F_s}} = 2 - \sqrt{2} \sin\left(\frac{n}{280}\,\frac{280\pi}{4} - \frac{\pi}{4}\right)$$
$$= 2 - \sqrt{2} \sin\left(70\pi t - \frac{\pi}{4}\right)\Big|_{t=\frac{n}{F_s}},$$

which leads to $y_c(t) = 2 - \sqrt{2} \sin\left(70\pi t - \frac{\pi}{4}\right)$.

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3.

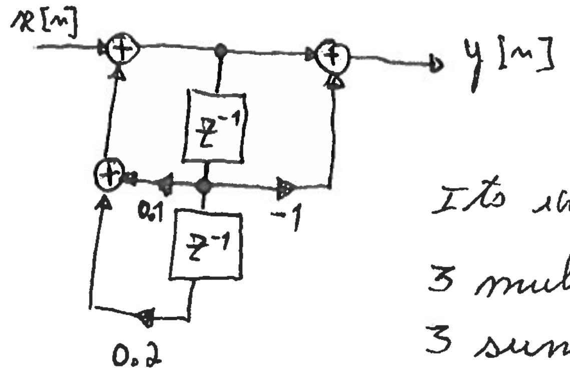
a) $y[n] = x[n] - x[n-1] + 0.1\,y[n-1] + 0.2\,y[n-2]$

It follows directly from the difference equation that

$Y(z) = X(z) - z^{-1}X(z) + 0.1\,z^{-1}Y(z) + 0.2\,z^{-2}Y(z)$ and, therefore:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - z^{-1}}{1 - 0.1\,z^{-1} - 0.2\,z^{-2}}, \text{ causal}$$

A possible realization structure is a direct type 2:



Its implementation cost is:
3 multiplications per output sample
3 sums per output sample
3 memory positions for coefficients
2 memory positions for data

b) Since $x[n] = u[n]$ then $X(z) = \frac{1}{1 - z^{-1}}$, $|z| > 1$ and

$$Y(z) = H(z)X(z) = \frac{1}{1 - z^{-1}} \cdot \frac{1 - z^{-1}}{1 - 0.1\,z^{-1} - 0.2\,z^{-1}} = \frac{1}{1 - 0.1\,z^{-1} - 0.2\,z^{-2}}, \text{ causal}$$

Finding the poles:

$$z^2 - 0.1\,z - 0.2 = 0 \quad \therefore \quad z = \frac{0.1 \pm \sqrt{0.01 + 0.8}}{2} = \begin{cases} 0.5 \\ -0.4 \end{cases}$$

makes it possible to write:

$$Y(z) = \frac{1}{(1 - 0.5\,z^{-1})(1 + 0.4\,z^{-1})}, \quad |z| > 0.5$$

$$= \frac{A}{1 - 0.5\,z^{-1}} + \frac{B}{1 + 0.4\,z^{-1}}$$

where

$$A = (1 - 0.5\,z^{-1})\,Y(z)\Big|_{z=0.5} = \frac{1}{1 + \frac{0.4}{0.5}} = \frac{5}{9}$$

$$B = (1 + 0.4\,z^{-1})\,Y(z)\Big|_{z=-0.4} = \frac{4}{9}$$

which leads to : $Y(z) = \underbrace{\frac{5/9}{1-0.5z^{-1}}}_{|z|>0.5} + \underbrace{\frac{4/9}{1+0.4z^{-1}}}_{|z|>0.4}$

and taking the inverse Z-Transform:

$$y[n] = \frac{5}{9}\, 0.5^{n} u[n] + \frac{4}{9}(-0.4)^{n} u[n]$$

c) Replacing z^{-1} in the delay chain by z^{-2} is the same as performing the same replacement in the transfer function:

$$H(z) = \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.4z^{-1})} \xrightarrow{z^{-1}\to z^{-2}} H(z^{2}) = \frac{1-z^{-2}}{(1-0.5z^{-2})(1+0.4z^{-2})}$$

$$= \frac{(1-z^{-1})(1+z^{-1})}{(1-\sqrt{0.5}\,z^{-1})(1+\sqrt{0.5}\,z^{-1})(1-j\sqrt{0.4}\,z^{-1})(1+j\sqrt{0.4}\,z^{-1})}, \quad |z| > \sqrt{0.5}$$

Thus, the implications are:

- the system order increases by a factor of 2 (it becomes 4)
- a positive zero gives rise to two zeros, one positive, another negative
- a positive pole gives rise to two poles, one positive, and another negative; their radii is the square root of that of the initial pole
- a negative pole gives rise to two poles, one imaginary and positive, and another imaginary and negative their radii is the square root of that of the initial pole
- as $H(z^{2})\big|_{z=e^{j\omega}} = H(e^{j2\omega})$ then, relative to $H(e^{j\omega})$, the new frequency response consists of the initial frequency response compressed by a factor of two; this is valid for both phase and magnitude parts of the frequency response.

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4.

a)
```
1: x = [1 0 3 0 5 0];
2: X = fft(x); N = length(X);
3: Y = zeros(size(X)); Y(1) = X(1);
4: Y(N:-1:2) = X(2:N);
5: Z = X.*Y; ifft(Z)
6: K = [0:N-1]; W = X.*(1+cos(K*2*pi/N)); ifft(W)
```

Lines 1 and 2 of the code set $x[n]$ and $X[K] = DFT\{x[n]\}$

Lines 3 and 4 set $Y[K] = X[(-K)_N] = X[N-K]$, $K = 1, \dots, N-1$

Line 5 makes $Z[K] = X[K] \cdot Y[K] = X[K] \cdot X[(-K)_N]$

and computes $z[n] = IDFT\{Z[K]\}$

Using the DFT properties:

$$x[n] \xleftrightarrow{DFT} X[K]$$

$$x[(-n)_N] \longleftrightarrow X[(-K)_N]$$

$$z[n] = x[n] \circledast x[(-n)_N] \longleftrightarrow X[K] \cdot X[(-K)_N] = Z[K]$$

which leads to

	..0	K=0 ↓ 1	0	3	0	5	K=N-1 ↓ 0	1 0 ....
$x[K]$		1	0	3	0	5	0	
$x[(-K)_N]$		1	0	5	0	3	0	
$x[-(-K)_N] = x[K]$		1	0	3	0	5	0	$\sum_K x[K]\,x[K] = 35$
$x[K-1]$		0	1	0	3	0	5	$\sum_K x[K]\,x[(K-1)] = 0$
$x[K-2]$		5	0	1	0	3	0	$\sum_K x[K]\,x[(K-2)] = 23$
$x[K-3]$		0	5	0	1	0	3	$\sum_K x[K]\,x[(K-3)] = 0$
$x[K-4]$		3	0	5	0	1	0	$\sum_K x[K]\,x[(K-4)] = 23$
$x[K-5]$		0	3	0	5	0	1	$\sum_K x[K]\,x[(K-5)] = 0$

Thus, $z[n] = x[n] \circledast x[(-n)_N] = \sum_{K=0}^{N-1} x[K]\,x[-(n-K)_N] = \sum_{K=0}^{N-1} x[K]\,x[(K-n)_N] =$

$= [35 \quad 0 \quad 23 \quad 0 \quad 23 \quad 0]$

b) Line 6 of the code sets:

$$w[K] = X[K] \cdot (1 + \cos(K \cdot 2\pi/N))$$
$$= X[K] \cdot \left(1 + \frac{e^{jK\frac{2\pi}{N}} + e^{-jK\frac{2\pi}{N}}}{2}\right)$$

$$= X[K] + \frac{1}{2} e^{jK\frac{2\pi}{N}} X[K] + \frac{1}{2} e^{-jK\frac{2\pi}{N}} X[K]$$

and, given the DFT property: $x[(n-n_0)_N] \longleftrightarrow e^{-jK\frac{2\pi}{N}n_0} X[K]$

we have: $x[(n-1)_N] \longleftrightarrow e^{-jK\frac{2\pi}{N}} X[K]$

$x[(n+1)_N] \longleftrightarrow e^{jK\frac{2\pi}{N}} X[K]$

which leads to:

$$w[n] = x[n] + \frac{1}{2} x[(n+1)_N] + \frac{1}{2} x[(n-1)_N]$$

c) as

	n=0					n=N-1
$x[n]$ ...0	1	0	3	0	5	0 | 1 0 ...
$x[(n+1)_N]$	0	3	0	5	0	1
$x[(n-1)_N]$	0	1	0	3	0	5

we obtain:

$$w[n] = [1\ 2\ 3\ 4\ 5\ 3]$$

5.

a) As a general concept, if $x[n]$ is a discrete-time signal, its DFT, $X[K]$, is N-periodic, i.e. $X[K] = X[K + lN]$, $\forall l \in \mathbb{Z}$
In the discrete-frequency domain,
N represents the period of $X[K]$, but also represents the sampling frequency in that domain, and $\frac{N}{2}$ represents the Nyquist frequency (half of the sampling frequency,

On the other hand, if $x[n]$ is real-valued, then, according to the DFT properties, $X[K] = X^*[N-K]$, $K=1,\dots,N-1$

$$= X^*[(-K)_N]$$

which means that the $X[K]$ sequence is conjugate symmetric i.e. the negative frequency axis is the conjugate mirror of the positive frequency axis. Considering the N-periodicity, this also means that when $x[n]$ is real-valued, $|X[K]|$ is mirrored with respect to the Nyquist frequency.

As in our case the sampling frequency is 8KHz, and the Nyquist frequency is 4KHz, real-valued signal components are easily identified by looking at even symmetries around 4KHz (for the same time). In this context, two real-valued signal components are identified:

- one having a constant frequency of 2KHz and existing between $t=0.1$s and $t=0.7$s
- another having a variable frequency around the mean frequency of 2KHz and existing between $t=0.2$s and $t=0.6$s.

The spectrogram also reveals two complex-valued signal components (two complex sinusoids):

- one whose frequency increases linearly between 0Hz and the sampling frequency (8KHz) from $t=0$s till $t=1.5$s
- another exhibiting a variable frequency around the mean frequency of 4KHz and existing between $t=1$s and $t=1.4$s.

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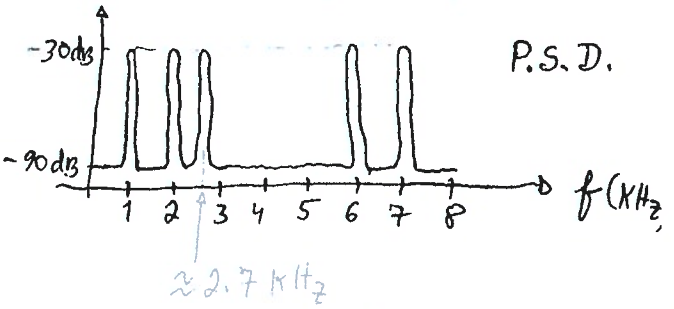
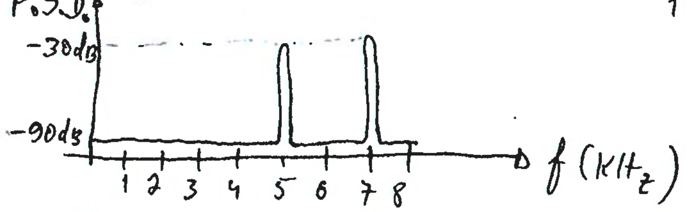
Spectrogram A has been obtained using the rectangular window because the fixed frequency of 2 KHz is represented by means of a thinner line than represented in spectrogram B (as a conseque of the wider main lobe width of the frequency response of the Hanning window), and because the spectrogram B is much cleaner than the spectrogram A, which is a consequence of the fact that the far-end leakage due to the Hanning window is much lower than in the case of the rectangular window.

b) In the case of $t = 0.5$ s, and taking a vertical line in the spectrogram (A or B, but B is easier), a plausible periodogram results as:



In the case of $t = 1.3$ s a plausible spectrogram is:



NOTE: only the case $t = 0.5$ was considered for a 100% correct answer.

c) As a result of the answer in a), we have:

- one narrow-band and real-valued signal component having a constant frequency of 2KHz and existing between $t=0.1$ and $t=0.7$
- one narrow-band and real-valued signal component whose frequency varies sinusoidally around $f = 2$ KHz, the frequency deviation is around 1 Hz, the period of the variation is $0.6 - 0.2 = 0.4$ s, and this signal exists between $t = 0.2$ s and $t = 0.6$ s.
- one narrow-band and complex-valued signal component whose center frequency increases for $t \in [0, 1.5]$ s between 0Hz an 8KHz
- one narrow-band and complex-valued signal component whose center frequency is 4Kz and varies sinusoidally for $t \in [1, 1.4]$ s, the frequency deviation is around 1KHz and period is 0.4 s.

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