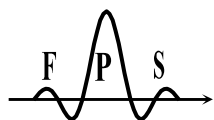


Summary

- *The DTFT of the auto-correlation and of the cross-correlation*
(extended review, week 02)
 - *the DTFT of the auto-correlation*
 - *the DTFT of the cross-correlation*
 - *auto/cross-correlation basic properties*
 - *auto/cross-correlation between output and input of an LSI system*
 - *examples*
- *The Z-Transform of the auto/cross-correlation*
(extended review, weeks 04/05)
 - *the Z-Transform of the auto-correlation*
 - *the Z-Transform of the cross-correlation*
- *Computing the auto/cross-correlation of finite-length sequences using the DFT*
 - *our starting point*
 - *the cross-correlation using the DFT*
 - *the auto-correlation using the DFT*



The DTFT of the auto-correlation and of the cross-correlation

- the DTFT of the auto-correlation

the auto-correlation is defined as (in this discussion, we admit energy signals)

$$r_x[\ell] = x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k] x^*[k - \ell]$$

it characterizes the similarity between a sequence and a copy of itself when it is shifted by a lag (ℓ)

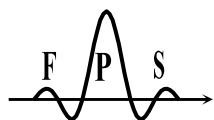
considering the DTFT properties

$$\begin{aligned} x[\ell] &\xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \\ x^*[\ell] &\xleftrightarrow{\mathcal{F}} X^*(e^{-j\omega}) \\ x[-\ell] &\xleftrightarrow{\mathcal{F}} X(e^{-j\omega}) \\ x^*[-\ell] &\xleftrightarrow{\mathcal{F}} X^*(e^{j\omega}) \end{aligned}$$

then

$$r_x[\ell] = x[\ell] * x^*[-\ell] \xleftrightarrow{\mathcal{F}} R_x(e^{j\omega}) = X(e^{j\omega}) \cdot X^*(e^{j\omega}) = |X(e^{j\omega})|^2$$

where $R_x(e^{j\omega}) = |X(e^{j\omega})|^2$ is called the spectral density of energy



The DTFT of the auto-correlation and of the cross-correlation

- the DTFT of the auto-correlation (cont.)
 - the Wiener-Khintchine Theorem: the auto-correlation and the spectral density of energy form a Fourier pair

$$r_x[\ell] \xleftrightarrow{\mathcal{F}} R_x(e^{j\omega}) = |X(e^{j\omega})|^2$$

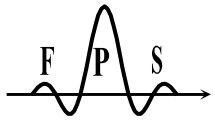
thus,

$$r_x[\ell] = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(e^{j\omega}) e^{j\omega\ell} d\omega$$

and, in particular, the energy of the signal can be found using

$$E = r_x[0] = \sum_{k=-\infty}^{+\infty} |x[k]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} R(e^{j\omega}) d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$$

which reflects the Parseval Theorem



The DTFT of the auto-correlation and of the cross-correlation

- the DTFT of the cross-correlation

the cross-correlation is defined as (we admit energy signals)

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k] y^*[k - \ell]$$

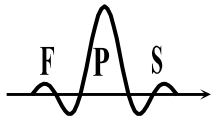
it characterizes the similarity between a sequence and a copy of another sequence when it is shifted by a lag (ℓ)

considering the DTFT properties

$$\begin{aligned} x[\ell] &\xleftrightarrow{\mathcal{F}} X(e^{j\omega}) \\ y[\ell] &\xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) \\ y^*[\ell] &\xleftrightarrow{\mathcal{F}} Y^*(e^{-j\omega}) \\ y[-\ell] &\xleftrightarrow{\mathcal{F}} Y(e^{-j\omega}) \\ y^*[-\ell] &\xleftrightarrow{\mathcal{F}} Y^*(e^{j\omega}) \end{aligned}$$

then

$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \xleftrightarrow{\mathcal{F}} R_{xy}(e^{j\omega}) = X(e^{j\omega}) \cdot Y^*(e^{j\omega})$$



The DTFT of the auto-correlation and of the cross-correlation

- auto/cross-correlation basic properties

- complex-conjugate symmetry (auto-correlation only!)

$$r_x[\ell] = r_x^*[-\ell] \qquad r_{xy}[\ell] = r_{yx}^*[-\ell]$$

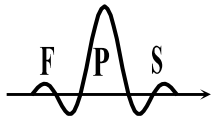
(implies that $R_x(e^{j\omega})$ is real-valued)

- upper bound

$$|r_x[\ell]| \leq r_x[0] \qquad |r_{xy}[\ell]| \leq \sqrt{r_x[0] \cdot r_y[0]}$$

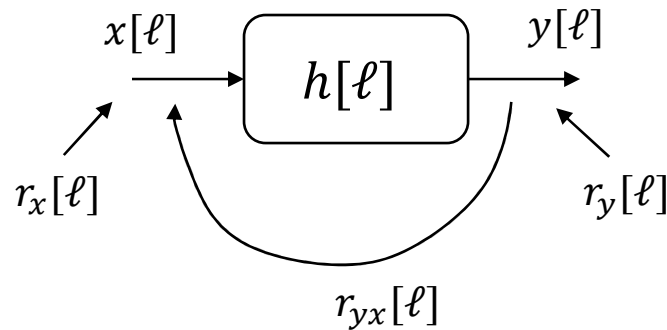
- normalized auto-correlation and cross-correlation

$$\rho_x[\ell] = \frac{r_x[\ell]}{r_x[0]}, \quad |\rho_x[\ell]| \leq 1 \qquad \rho_{xy}[\ell] = \frac{r_{xy}[\ell]}{\sqrt{r_x[0] \cdot r_y[0]}}, \quad |\rho_{xy}[\ell]| \leq 1$$



The DTFT of the auto-correlation and of the cross-correlation

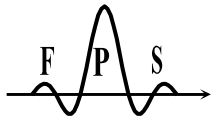
- auto/cross-correlation between output and input of an LSI system
 - the (quite important !) last two equations are stated without proof



$$y[\ell] = x[\ell] * h[\ell] \quad \xleftrightarrow{\mathcal{F}} \quad Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

$$r_{yx}[\ell] = r_x[\ell] * h[\ell] \quad \xleftrightarrow{\mathcal{F}} \quad R_{yx}(e^{j\omega}) = R_x(e^{j\omega})H(e^{j\omega})$$

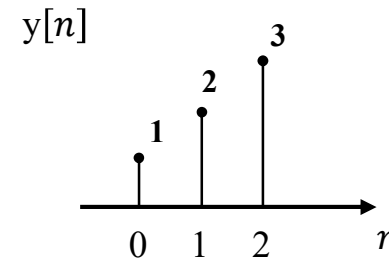
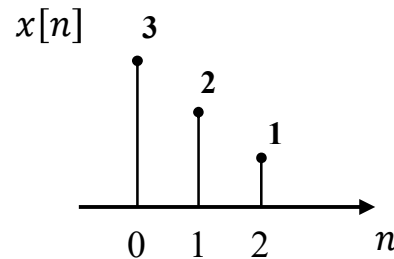
$$r_y[\ell] = r_x[\ell] * h[\ell] * h^*[-\ell] \quad \xleftrightarrow{\mathcal{F}} \quad R_y(e^{j\omega}) = R_x(e^{j\omega})|H(e^{j\omega})|^2$$



The DTFT of the auto-correlation and of the cross-correlation

- examples

let us admit two finite-length discrete-time signals, $x[n]$ and $y[n]$



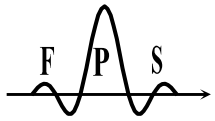
it can be easily concluded that

$$x[\ell] = 3\delta[\ell] + 2\delta[\ell - 1] + \delta[\ell - 2] \xleftrightarrow{\mathcal{F}} X(e^{j\omega}) = 3 + 2e^{-j\omega} + e^{-j2\omega}$$

$$y[\ell] = \delta[\ell] + 2\delta[\ell - 1] + 3\delta[\ell - 2] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = 1 + 2e^{-j\omega} + 3e^{-j2\omega}$$

$$R_x(e^{j\omega}) = 3e^{j2\omega} + 8e^{j\omega} + 14 + 8e^{-j\omega} + 3e^{-j2\omega} = R_y(e^{j\omega}), \text{ (why ?)}$$

$$R_{xy}(e^{j\omega}) = 9e^{j2\omega} + 12e^{j\omega} + 10 + 4e^{-j\omega} + e^{-j2\omega}$$



The DTFT of the auto-correlation and of the cross-correlation

- examples

random sequences may exhibit $r_x[\ell] = \delta[\ell]$

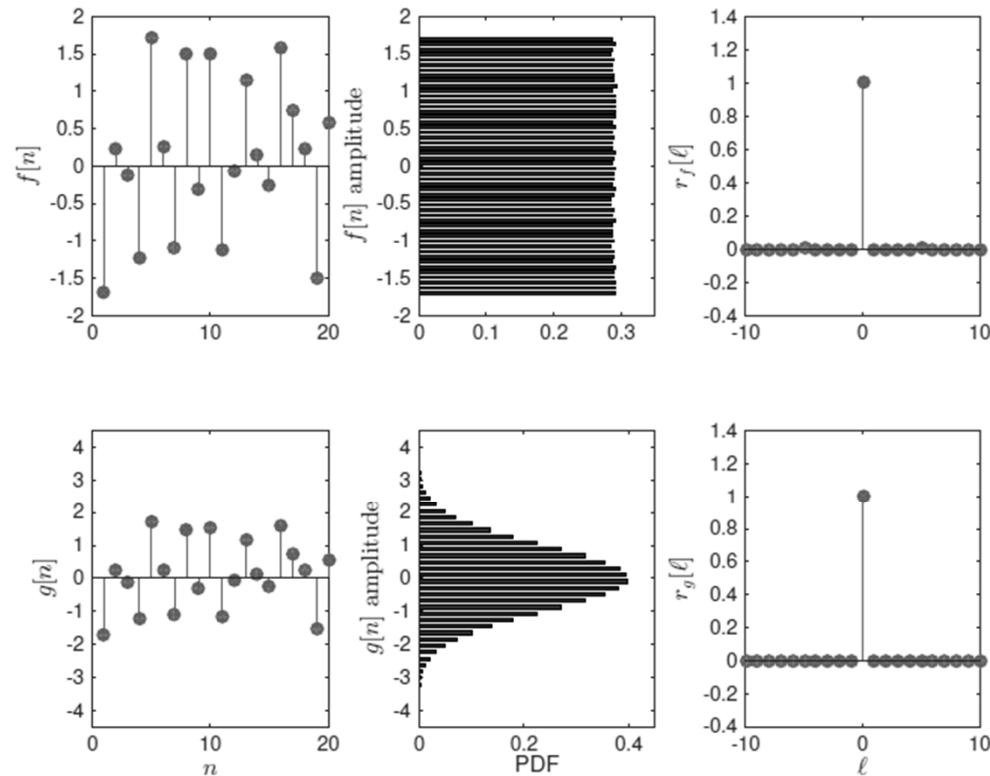
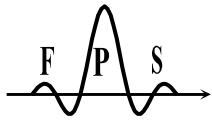


Fig. 1.84 From left to right, the top row of plots represents a few samples of a random sequence ($f[n]$), the flat PDF of its samples amplitudes, and its auto-correlation function $r_f[\ell]$. The bottom row represents the same sequence of plots in the case the random sequence ($g[n]$) has a Gaussian PDF. In both cases, the auto-correlation functions, $r_f[\ell]$ and $r_g[\ell]$, are identical and correspond to the unit impulse.



The DTFT of the auto-correlation and of the cross-correlation

- examples

deterministic sequences may exhibit $r_x[\ell] = \delta[\ell]$

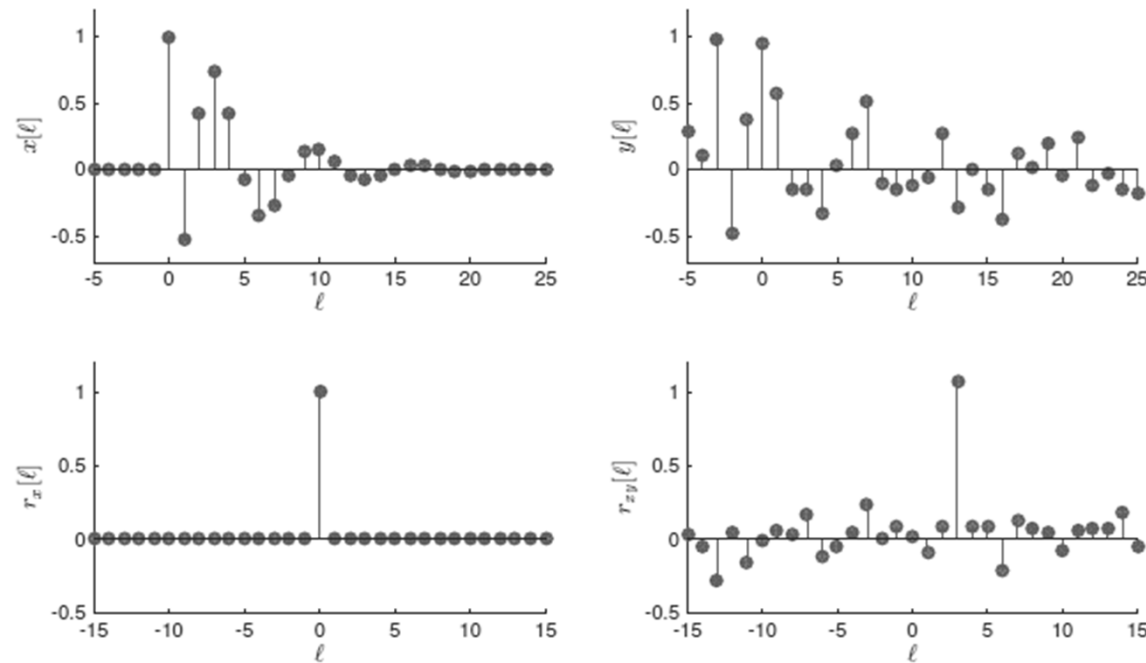
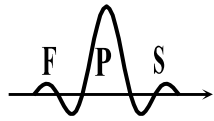


Fig. 1.76 Example of a waveform $x[\ell]$ (top left figure), its auto-correlation function $r_x[\ell]$ (bottom left figure), a waveform $y[\ell]$ consisting of noisy version of $x[\ell + 3]$ (top right figure), and the cross-correlation function $r_{xy}[\ell]$ (bottom right figure).



The DTFT of the auto-correlation and of the cross-correlation

- examples

the auto-correlation is useful to find the period of periodic signals: it is signalled by the *first* local maximum in the $\rho_x[\ell]$ or $\rho_x[\ell]$ functions ($\ell \neq 0$, why ?)

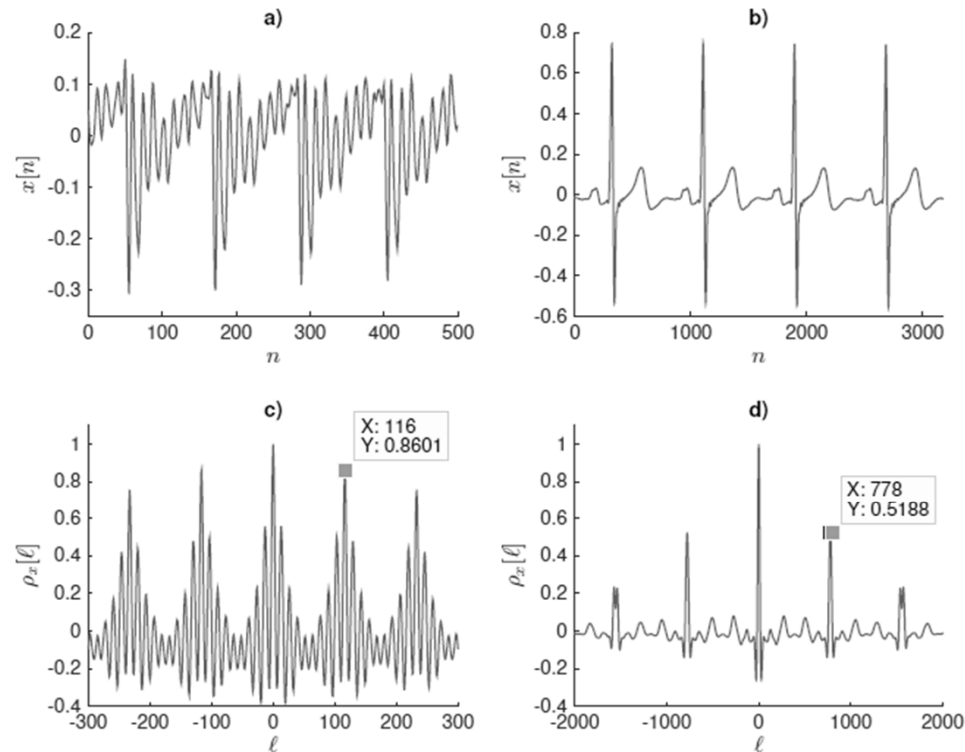
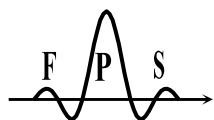


Fig. 1.80 Panel a) represents a segment of a periodic sequence ($x[n]$) corresponding to a vowel signal (the sampling frequency is 22050 Hz), and panel c) represents its normalized auto-correlation function ($\rho_x[\ell]$). Panel b) represents a segment of an ECG signal that has been captured using 1 kHz as the sampling frequency. Panel c) represents its normalized auto-correlation function (see text for details on the auto-correlation analysis).



The Z-Transform of the auto/cross-correlation

- the Z-Transform of the auto-correlation

the auto-correlation is defined as (in this discussion, we admit energy signals)

$$r_x[\ell] = x[\ell] * x^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]x^*[k - \ell]$$

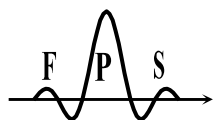
considering the Z-Transform properties

$$\begin{aligned} x[\ell] &\xleftrightarrow{Z} X(z), & RoC = R_x \equiv r_E < |z| < r_D \\ x^*[\ell] &\xleftrightarrow{Z} X^*(z^*), & RoC = R_x \\ x[-\ell] &\xleftrightarrow{Z} X(z^{-1}), & RoC = 1/R_x \equiv 1/r_D < |z| < 1/r_E \\ x^*[-\ell] &\xleftrightarrow{Z} X^*(1/z^*), & RoC = 1/R_x \end{aligned}$$

then

$$r_x[\ell] = x[\ell] * x^*[-\ell] \xleftrightarrow{Z} R_x(z) = X(z) \cdot X^*(1/z^*), \quad RoC = R_x \cap 1/R_x$$

where $R_x(z) = X(z) \cdot X^*(1/z^*)$ is called the energy spectrum



The Z-Transform of the auto/cross-correlation

- the Z-Transform of the auto-correlation (cont.)
 - the Wiener-Khintchine Theorem: the auto-correlation and the energy spectrum form a Z-Transform pair

$$r_x[\ell] \xleftrightarrow{Z} R_x(z) = X(z) \cdot X^*(1/z^*)$$

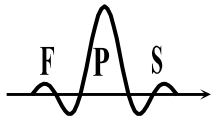
thus,

$$r_x[\ell] = \frac{1}{2\pi j} \oint_C R_x(z) Z^{\ell-1} dz$$

and, in particular, the energy of the signal can be found using

$$E = r_x[0] = \sum_{k=-\infty}^{+\infty} |x[k]|^2 = \frac{1}{2\pi j} \oint_C X(z) \cdot X^*(1/z^*) Z^{-1} dz$$

which reflects the Parseval Theorem in the Z-domain



The Z-Transform of the auto/cross-correlation

- the Z-Transform of the cross-correlation
the cross-correlation is defined as (we admit energy signals)

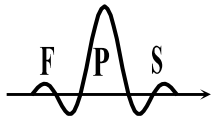
$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k] y^*[k - \ell]$$

considering the Z-Transform properties

$$\begin{aligned} x[\ell] &\xleftrightarrow{Z} X(z), & RoC = R_x \\ y[\ell] &\xleftrightarrow{Z} Y(z), & RoC = R_y \\ y^*[\ell] &\xleftrightarrow{Z} Y^*(z^*), & RoC = R_y \\ y[-\ell] &\xleftrightarrow{Z} Y(z^{-1}), & RoC = 1/R_y \\ y^*[-\ell] &\xleftrightarrow{Z} Y^*(1/z^*), & RoC = 1/R_y \end{aligned}$$

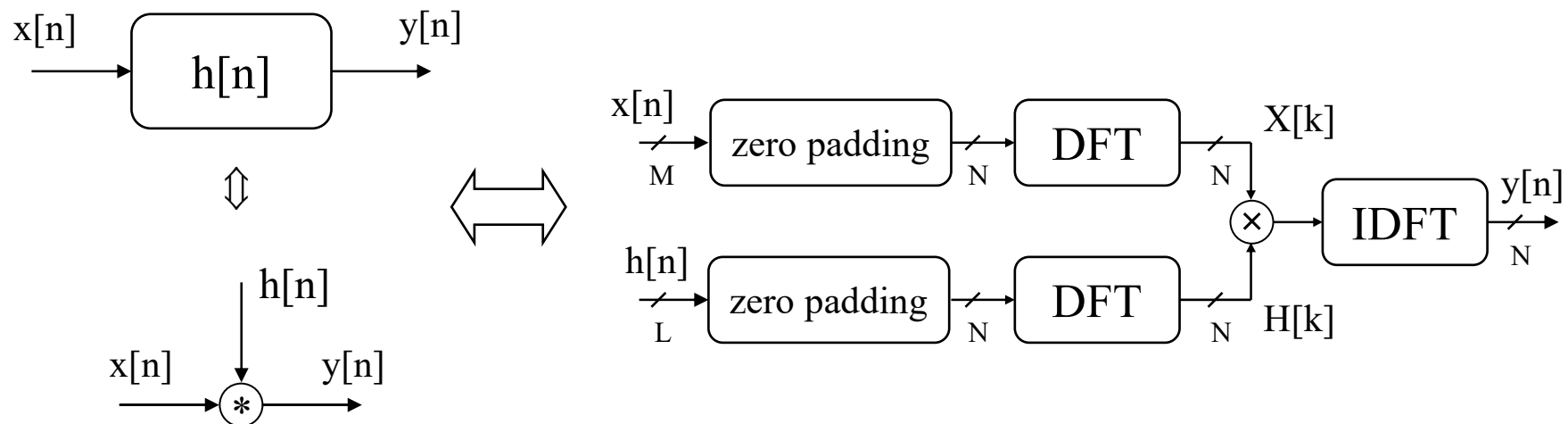
then

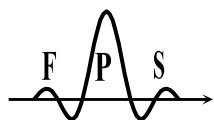
$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \xleftrightarrow{Z} R_{xy}(z) = X(z) \cdot Y^*(1/z^*), \quad RoC = R_x \cap 1/R_y$$



Computing the AC/CC of finite-length sequences using the DFT

- our starting point
 - we have seen that if a sequence $x[n]$ has length M , and another sequence $h[n]$ has length L , the linear convolution between them corresponds to a sequence whose length is $L+M-1$
 - we also have seen that if the signals are zero-padded and made periodic with period N , then the linear result convolution result may also be found using the DFT and its properties as long as $N \geq L+M-1$
 - in this case, the circular convolution yields the same result of the linear convolution according to the following block diagram

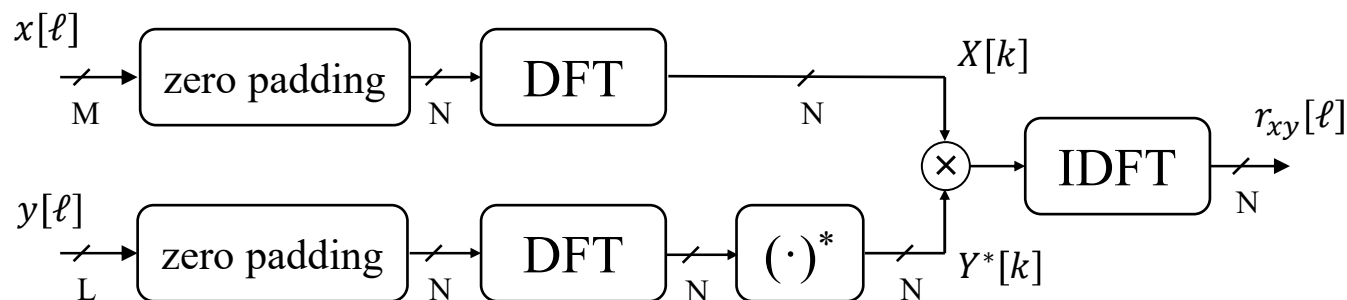


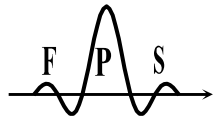


Computing the AC/CC of finite-length sequences using the DFT

- the cross-correlation using the DFT
 - assuming that both sequences, $x[\ell]$ and $y[\ell]$, are suitably zero-padded such that the circular convolution reduces to the linear convolution, then $r_{xy}[\ell]$ can be computed using DFT-based frequency domain processing

$$\begin{array}{lll}
 x[\ell] & \xleftrightarrow{DFT} & X[k] \\
 y[\ell] & \xleftrightarrow{DFT} & Y[k] \\
 y^*[\ell] & \xleftrightarrow{DFT} & Y^*[(-k)_N] \\
 y[(-\ell)_N] & \xleftrightarrow{DFT} & Y[(-k)_N] \\
 y^*[(-\ell)_N] & \xleftrightarrow{DFT} & Y^*[k] \\
 r_{xy}[\ell] = x[\ell] * y^*[-\ell] & \xleftrightarrow{DFT} & R_{xy}[k] = X[k] \cdot Y^*[k]
 \end{array}$$





Computing the AC/CC of finite-length sequences using the DFT

- the auto-correlation using the DFT
 - this case is a particular case in the sense that $x[\ell] = y[\ell]$ in the previous slide, which leads to the following (simplified) relationships and block diagram that yields $r_x[\ell]$

$$\begin{array}{rcl}
 x[\ell] & \xleftrightarrow{DFT} & X[k] \\
 x^*[\ell] & \xleftrightarrow{DFT} & X^*[-k]_N \\
 x[-\ell]_N & \xleftrightarrow{DFT} & X[k]_N \\
 x^*[-\ell]_N & \xleftrightarrow{DFT} & X^*[k] \\
 r_x[\ell] = x[\ell] * x^*[-\ell] & \xleftrightarrow{DFT} & R_x[k] = |X[k]|^2
 \end{array}$$

