FPS, WEEK 7

PROB 1

a) $x[n] \equiv$ number of new students in year n

$y[n] \equiv$ total number of students in year n

$70\% \equiv$ success rate

Thus, according to the problem statement, the difference equation results as:

$$y[n] = x[n] + (1-0.7)\,y[n-1]$$

in words: each year the total number of students includes the new students and 30% of the student population enrolled in the course, the previous year.

b) If $x[n] = 250\,u[n] \xleftrightarrow{\mathcal{Z}} X(z) = \dfrac{250}{1-z^{-1}}, \ |z|>1$

The system transfer function results from the Z-Transform of the difference equation:

$$Y(z) = X(z) + 0.3\,z^{-1}Y(z) \Leftrightarrow$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1-0.3\,z^{-1}}, \quad |z| > 0.3$$

Hence:

$$Y(z) = H(z)\,X(z) = \frac{250}{(1-z^{-1})(1-0.3z^{-1})}, \quad |z|>1$$

Finding the inverse Z-Transform:

$$Y(z) = \frac{A}{1-z^{-1}} + \frac{B}{1-0.3\,z^{-1}}$$

with: $A = \left.\dfrac{250}{1-\frac{3}{10}z^{-1}}\right|_{z=1} = \dfrac{2500}{7} =$

and $B = \left.\dfrac{250}{1 - z^{-1}}\right|_{z = \frac{3}{10}} = \dfrac{250}{1 - \frac{10}{3}} = -\dfrac{750}{7}$

which leads to:

$$Y(z) = \underbrace{\frac{2500/7}{1 - z^{-1}}}_{|z|>1} - \underbrace{\frac{750/7}{1 - 0.3z^{-1}}}_{|z|>0.3}$$

whose inverse Z-Transform is:

$$y[n] = \frac{2500}{7} u[n] - 0.3^{n} \, \frac{750}{7} \, u[n]$$

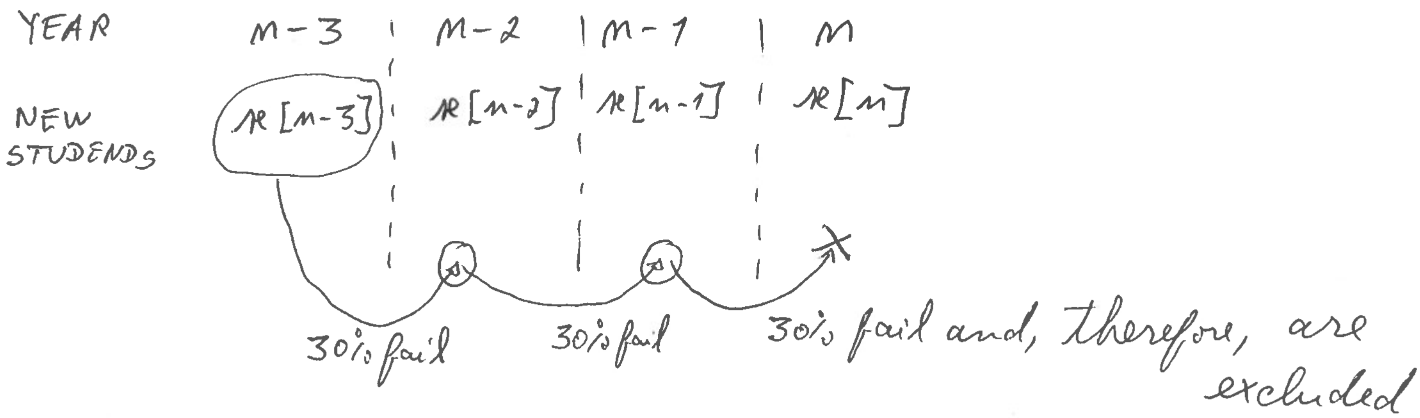
Now, when n gets very large, the population tends to

$$\lim_{n \to +\infty} y[n] = \frac{2500}{7} \simeq 357 \text{ students}$$

Or, using the final value theorem:

$$\lim_{n \to +\infty} y[n] = \lim_{z \to 1} (1 - z^{-1}) Y(z) = \frac{250}{1 - \frac{3}{10}} = \frac{2500}{7} \simeq 357$$

e) In this question, we have to exclude those students who would fail for the third time; in order to better understand this, it is convenient to use a graphical representation:



Thus, the new difference equation results as:

$$y[n] = x[n] + 0.3\,y[n-1] - 0.3^3\,x[n-3]$$

In the Z-domain:

$$Y(z) = X(z) + 0.3\,z^{-1}Y(z) - 0.3^3\,z^{-3}X(z)$$

$$\therefore \quad \frac{Y(z)}{X(z)} = \frac{1-(0.3z^{-1})^3}{1-0.3z^{-1}}, \quad |z| > 0.3$$

and... we have seen this kind of polynomial division already!

$$\begin{array}{rrrr|l}
-(0.3z^{-1})^3 & & & +1 & -0.3z^{-1}+1 \\
\cline{5-5}
+(0.3z^{-1})^3 & -(0.3z^{-1})^2 & & & (0.3z^{-1})^2 + 0.3z^{-1} + 1 \\
\cline{1-2}
0 & -(0.3z^{-1})^2 & & & \\
 & +(0.3z^{-1})^2 & -0.3z^{-1} & & \\
\cline{2-3}
 & 0 & -0.3z^{-1} & +1 & \\
 & & +0.3z^{-1} & -1 & \\
\cline{3-4}
 & & 0 & 0 &
\end{array}$$

So,

$$\frac{Y(z)}{X(z)} = 1 + 0.3z^{-1} + (0.3z^{-1})^2, \quad |z| > 0$$

In words, this means that, each year, the total number of students includes new students, those who failed once, and those who failed twice

$$\text{Now: } Y(z) = H(z)X(z) = \left(1 + 0.3z^{-1} + 0.3^2z^{-2}\right)\frac{250}{1-z^{-1}}, \quad |z|>1$$

$$= \underset{|z|>1}{\frac{250}{1-z^{-1}}} + 0.3\,\underset{|z|>1}{\frac{z^{-1}}{1-z^{-1}}} \times 250 + 0.09\,\underset{|z|>1}{\frac{z^{-2}}{1-z^{-1}}} \times 250$$

which means that $y[n] = 250\,u[n] + 75\,u[n-1] + 22.5\,u[n-2]$

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and, finally:

$$\lim_{n\to+\infty} y[n] = 250 + 75 + 22.5 = 347.5 \text{ students}$$

PROB 2

Just solution topics:

$x[n] \equiv$ monthly payment

$y[n] \equiv$ accumulated balance: we assume $y[-1] = 0$

$$y[n] = x[n] + y[n-1] + \frac{0.03}{12} y[n-1] = x[n] + \frac{12.03}{12} y[n-1]$$

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1}{1 - \frac{12.03}{12} z^{-1}}, \quad |z| > \frac{12.03}{12} \qquad \text{NOTE: } \frac{12.03}{12} > 1$$

If $x[n] = P\,u[n]$ $\quad \therefore \quad X(z) = \frac{P}{1 - z^{-1}}, \quad |z| > 1$

$$Y(z) = H(z)X(z) = \frac{P}{(1 - z^{-1})\left(1 - \frac{12.03}{12} z^{-1}\right)} = \underbrace{\frac{A}{1 - z^{-1}}}_{|z|>1} + \underbrace{\frac{B}{1 - \frac{12.03}{12} z^{-1}}}_{|z| > \frac{12.03}{12}}$$

$$A = -\frac{12P}{0.03}$$

$$B = \frac{12.03P}{0.03}$$

$$\therefore \quad y[n] = A\,u[n] + B\left(\frac{12.03}{12}\right)^n u[n]$$

$$50\,000 = P\left[\left(\frac{12.03}{12}\right)^{240} \frac{12.03}{0.03} - \frac{12}{0.03}\right] \iff P = 151{,}46 \text{ €}$$