L.EEC
L.EEC025 - Fundamentals of Signal Processing

Formulae Sheet

Laplace Transform

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st}\,dt \qquad x(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

$$e^{-\alpha t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+\alpha}, \qquad \Re\{s\} > -\alpha$$

Discrete-time convolution

$$x[n]*h[n] = \sum_{\ell=-\infty}^{+\infty} x[\ell]h[n-\ell] = \sum_{\ell=-\infty}^{+\infty} h[\ell]x[n-\ell] = h[n]*x[n]$$

Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} \qquad x[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\omega})e^{j\omega n}\,d\omega$$

$$x[-n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{-j\omega}) \qquad x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$$

$$x_e[n] = \frac{x[n]+x^*[-n]}{2} = x_e^*[-n] \overset{\mathcal{F}}{\longleftrightarrow} \Re\{X(e^{j\omega})\}$$

$$x_o[n] = \frac{x[n]-x^*[-n]}{2} = -x_o^*[-n] \overset{\mathcal{F}}{\longleftrightarrow} j\Im\{X(e^{j\omega})\}$$

$$\Re\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} X_e\left(e^{j\omega}\right) = X_e^*\left(e^{-j\omega}\right)$$

$$j\Im\{x[n]\} \overset{\mathcal{F}}{\longleftrightarrow} X_o\left(e^{j\omega}\right) = -X_e^*\left(e^{-j\omega}\right)$$

$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}X\left(e^{j\omega}\right) \qquad e^{jn\omega_0}x[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j(\omega-\omega_0)}\right)$$

$$nx[n] \overset{\mathcal{F}}{\longleftrightarrow} j\frac{dX\left(e^{j\omega}\right)}{d\omega} \qquad x[n]*y[n] \overset{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right)Y\left(e^{j\omega}\right)$$

(periodic convolution)

$$x[n]y[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{+\pi} X\left(e^{j\theta}\right)Y\left(e^{j(\omega-\theta)}\right)d\theta$$

$$e^{jn\omega_0} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\sum_{k=-\infty}^{+\infty}\delta(\omega-\omega_0+k2\pi)$$

$$\sum_{\ell=-\infty}^{+\infty}\delta[n-\ell] \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\sum_{k=-\infty}^{+\infty}\delta(\omega+k2\pi)$$

$$\delta[n] \overset{\mathcal{F}}{\longleftrightarrow} 1 \qquad \delta[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}$$

$$a^n u[n], \quad |a|<1 \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-ae^{-j\omega}}$$

$$(n+1)a^n u[n], \quad |a|<1 \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|1-ae^{-j\omega}|^2}$$

Low-pass filter

$$\frac{\sin(n\omega_c)}{n\pi} \overset{\mathcal{F}}{\longleftrightarrow} \begin{cases}1, & |\omega| \le \omega_c\\ 0, & \omega_c < |\omega| \le \pi\end{cases}$$

Parseval Theorem

$$\sum_{n=-\infty}^{+\infty} x[n]y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)d\omega$$

Cross-correlation

$$r_{xy}[\ell] = x[\ell]*y^*[-\ell] = \sum_{k=-\infty}^{+\infty} x[k]\,y^*[k-\ell]$$

$$r_{xy}[\ell] \overset{\mathcal{F}}{\longleftrightarrow} R_{xy}\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)Y^*\left(e^{j\omega}\right)$$

$$r_{xy}[\ell] = r_{yx}^*[-\ell] \qquad \left|r_{xy}[\ell]\right| \le \sqrt{r_x[0]r_y[0]}$$

Auto-correlation
$$r_x[\ell] = x[\ell] * x^*[-\ell] \overset{\mathcal{F}}{\longleftrightarrow} R_x(e^{j\omega}) = |X(e^{j\omega})|^2$$
$$r_x[\ell] = r_x^*[-\ell] \qquad |r_x[\ell]| \le r_x[0]$$

Ideal sampling
$$x_c(t) \overset{\mathcal{F}}{\longleftrightarrow} X_c(\Omega) \qquad x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$
$$x[n] = x_c(nT) \longrightarrow X(e^{j\omega}) = \frac{1}{T}\sum_{k=-\infty}^{+\infty} X_c\left(\Omega - \mathrm{k}\frac{2\pi}{T}\right), \quad \omega = \Omega \mathrm{T}$$

Ideal reconstruction
$$y_c(t) = \sum_{n=-\infty}^{+\infty} y[n]\,\mathrm{sinc}\left(\frac{\pi}{T}(t-nT)\right), \quad \mathrm{sinc}(\theta) = \frac{\sin(\theta)}{\theta}$$

Z-Transform
$$X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}, \; RoC \equiv C_x \qquad x[n] = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

$$x[n-n_0] \overset{z}{\longleftrightarrow} z^{-n_0}X(z), \; RoC \equiv C_x$$
$$z_0{}^n x[n] \overset{z}{\longleftrightarrow} X(z/z_0), \; RoC \equiv |z_0|C_x$$
$$x^*[n] \overset{z}{\longleftrightarrow} X^*(z^*), \; RoC \equiv C_x$$
$$x[-n] \overset{z}{\longleftrightarrow} X(1/z), \; RoC \equiv 1/C_x$$
$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \; RoC \equiv C_x$$
$$x[n] * y[n] \overset{z}{\longleftrightarrow} X(z)Y(z), \; RoC \equiv C_x \cap C_y$$

(complex convolution)
$$x[n]y^*[n] \overset{z}{\longleftrightarrow} \frac{1}{2\pi j}\oint X(v)Y^*(z^*/v^*)v^{-1}dv, \; RoC \equiv C_x C_y$$

$$a^n u[n] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \; |z| > |a|$$
$$-a^n u[-n-1] \overset{z}{\longleftrightarrow} \frac{1}{1-az^{-1}}, \; |z| < |a|$$
$$na^n u[n] \overset{z}{\longleftrightarrow} \frac{az^{-1}}{(1-az^{-1})^2}, \; |z| > |a|$$
$$a^n \sin(n\omega_0)\,u[n] \overset{z}{\longleftrightarrow} \frac{a\sin(\omega_0)\,z^{-1}}{1-2a\cos(\omega_0)\,z^{-1}+a^2z^{-2}}, \; |z| > |a|$$
$$a^n \cos(n\omega_0)\,u[n] \overset{z}{\longleftrightarrow} \frac{1-a\cos(\omega_0)\,z^{-1}}{1-2a\cos(\omega_0)\,z^{-1}+a^2z^{-2}}, \; |z| > |a|$$

Parseval Theorem
$$\sum_{n=-\infty}^{+\infty} x[n]y^*[n] = \frac{1}{2\pi j}\oint X(z)Y^*(1/z^*)z^{-1}dz, \; RoC \equiv C_x C_y$$

Correlation in Z-domain
$$r_x[\ell] = x[\ell] * x^*[-\ell] \overset{z}{\longleftrightarrow} R_x(z) = X(z)X^*\left(\frac{1}{z^*}\right), \; RoC \equiv C_x \cap 1/C_x$$
$$r_{xy}[\ell] = x[\ell] * y^*[-\ell] \overset{z}{\longleftrightarrow} R_{xy}(z) = X(z)Y^*\left(\frac{1}{z^*}\right), RoC \equiv C_x \cap 1/C_y$$

Residues (1st-order poles)
$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1-d_k z^{-1}} \qquad A_k = (1-d_k z^{-1})X(z)|_{z=d_k}$$

Frequency Response and Group Delay
$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})} \qquad \tau(\omega) = -\frac{d}{d\omega}\{\angle H(e^{j\omega})\}$$

All-pass (gain normalized)
$$H(z) = \frac{z^{-1}-\alpha^*}{1-\alpha z^{-1}} \qquad H(z) = \frac{1}{|\alpha|}\frac{z-\frac{1}{\alpha^*}}{z-\alpha}$$

Linear-phase system
$$H(e^{j\omega}) = A(\omega)e^{j(\beta-\alpha\omega)}; \quad A(\omega) \text{ is real-valued;} \quad \beta, \alpha \text{ are constants}$$

IIR filter (causal)

$$y[n] = \sum_{k=0}^{M-1} b_k x[n-k] + \sum_{\ell=1}^{N-1} a_\ell y[n-\ell]$$

Butterworth filter

$$H_b(s)H_b(-s) = \frac{1}{1+\left(\frac{s}{\mathrm{j}\Omega_c}\right)^{2N}} \qquad |H_b(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

$$s_k = \Omega_c e^{j\frac{\pi}{2N}(2k+N+1)}, \quad k = 0,1,\ldots,2N-1$$

Impulse Invariance

$$h[n] = T h_c(nT) \qquad H\left(e^{j\omega}\right) = \sum_{k=-\infty}^{+\infty} H_c\left(\frac{\omega - k2\pi}{T}\right)$$

$$H_c(s) = \sum_{k=0}^{N-1} \frac{A_k}{s - s_k} \qquad H(z) = T\sum_{k=0}^{N-1} \frac{A_k}{1 - e^{s_k T} z^{-1}}$$

Bilinear Transformation

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \qquad z = \frac{2+sT}{2-sT}$$

$$\Omega = \frac{2}{T}\tan\left(\frac{\omega}{2}\right) \qquad \omega = 2\tan^{-1}\left(\frac{\Omega \mathrm{T}}{2}\right)$$

Rectangular window

$$w_R[n] = u[n] - u[N] = \begin{cases} 1, & n = 0,1,\ldots,N-1 \\ 0, & other\ n \end{cases}$$

Generalized Hamming window

$$w_H[n] = \left(\alpha - (1-\alpha)\cos\left(\frac{2\pi n}{N-1}\right)\right) w_R[n]$$

Hamming: $\alpha = 0.54$ Hanning: $\alpha = 0.50$

Window method (FIR)

$$H_d\left(e^{j\omega}\right) \overset{\mathcal{F}}{\longleftrightarrow} h_d[n] \qquad h[n] = h_d[n] w[n]$$

$$H\left(e^{j\omega}\right) = H_d\left(e^{j\omega}\right) * W\left(e^{j\omega}\right) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} H_d\left(e^{j\theta}\right) W\left(e^{j(\omega-\theta)}\right) d\theta$$

Parks-McClellan (Linear-phase FIR)

$H\left(e^{j\omega}\right) = A(\omega) e^{-j(\alpha\omega - \beta)}$; $A(\omega)$ is real-valued; β, α are constants

$$A(\omega) = Q(\omega)\sum_{n=0}^{r} a[n](\cos\omega)^n$$

$Q(\omega) = 1$: (FIR type 1) $Q(\omega) = \cos\left(\frac{\omega}{2}\right)$: (FIR type 2)

$Q(\omega) = \sin(\omega)$: (FIR type 3) $Q(\omega) = \sin\left(\frac{\omega}{2}\right)$: (FIR type 4)

Sampling the Fourier Transform

$$Y[k] = X(z)\big|_{z=e^{jk\frac{2\pi}{N}}} = X\left(e^{j\omega}\right)\big|_{\omega=k\frac{2\pi}{N}}, \quad k = 0,1,\ldots,N-1$$

$$x[n] \overset{Z}{\longleftrightarrow} X(z) \qquad y[n] = \sum_{\ell=-\infty}^{+\infty} x[n+\ell N]$$

Transfer function reconstruction

$$H(z) = \frac{1-z^{-1}}{N}\sum_{k=0}^{N-1}\frac{H[k]}{1 - W_N^{-k} z^{-1}}, \quad W = e^{-j2\pi}$$

DFT $\left(W_\beta^\alpha = e^{-j2\pi\frac{\alpha}{\beta}}\right)$

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

$$x[(-n)_N] \overset{DFT}{\longleftrightarrow} X[(-k)_N] \qquad x^*[n] \overset{DFT}{\longleftrightarrow} X^*[(-k)_N]$$

$$x_{ep}[n] = \frac{x[n] + x^*[(-n)_N]}{2} = x_{ep}^*[(-n)_N] \overset{DFT}{\longleftrightarrow} \Re\{X[k]\}$$

$$x_{op}[n] = \frac{x[n] - x^*[(-n)_N]}{2} = -x_{op}^*[(-n)_N] \overset{DFT}{\longleftrightarrow} j\Im\{X[k]\}$$

$$\Re\{x[n]\} \overset{DFT}{\longleftrightarrow} X_{ep}\left(e^{j\omega}\right)$$

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L.EEC025 - Fundamentals of Signal Processing

$$j\Im\{x[n]\} \overset{DFT}{\longleftrightarrow} X_{op}\left(e^{j\omega}\right)$$

$$x[(n-n_0)_N] \overset{DFT}{\longleftrightarrow} e^{-j\frac{2\pi}{N}kn_0}X[k]$$

$$e^{j\frac{2\pi}{N}nk_0}x[n] \overset{DFT}{\longleftrightarrow} X[(k-k_0)_N]$$

$$x[n] \circledast h[n] \overset{DFT}{\longleftrightarrow} X[k]H[k] \qquad x[n]h[n] \overset{DFT}{\longleftrightarrow} \frac{1}{N}X[k] \circledast H[k]$$

$$x[n] \circledast h[n] = \sum_{\ell=0}^{N-1} x[\ell]h[(n-\ell)_N]$$

Decimation $\quad d[n] = x[nM] \quad D(z) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(W_M^k z^{\frac{1}{M}}\right)$

Interpolation $\quad c[n] = \begin{cases} x\left[\frac{n}{L}\right], & n \text{ multiple of } L \\ 0, & \text{other } n \end{cases} \quad C(z) = X(z^L)$

FFT DIT $\quad X[k] = G[k] + W_N^k H[k] \quad X[k+N/2] = G[k] - W_N^k H[k]$

FFT DIF $\quad g[n] = x[n] + x[n+N/2] \quad h[n] = (x[n] - x[n+N/2])W_N^n$

Optimum Wiener filter $\quad \sum_{\ell=-\infty}^{+\infty} h_n[\ell]r_x[\ell-k] = r_{dx}[k] \quad r_{dx}[k] = d[k] * x^*[-k]$, $\quad r_x[k] = x[k] * x^*[-k]$

LMS adaptive filter

$$h_{n+1}[k] = h_n[k] + \mu x[n-k]e[n] \quad ; \quad 0 \le \mu \le \frac{1}{NP_x}$$

$$P_x = \frac{1}{M}\sum_{\ell=0}^{L-1}|x[\ell]|^2$$

Arithmetic/Geometric progression $\quad \sum_{k=1}^{M} k = \frac{M(M+1)}{2} \quad \sum_{k=0}^{M} \alpha^k = \frac{1-\alpha^{M+1}}{1-\alpha}$

Sum of squares $\quad \sum_{k=1}^{M} k^2 = \frac{M(M+1)(2M+1)}{6}$

Trigonometric identities

$$\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$$

$$\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha)$$

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{1}{2}(\alpha+\beta)\right)\cos\left(\frac{1}{2}(\alpha-\beta)\right)$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{1}{2}(\alpha+\beta)\right)\cos\left(\frac{1}{2}(\alpha-\beta)\right)$$

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\sin(\theta)$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1
$\cos(\theta)$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\tan(\theta)$	0	$\sqrt{3}/3$	1	$\sqrt{3}$	∞