Overview

- *The Discrete Fourier Transform*
  - *Concept*
  - *Review on the Fourier representation of signals*
    - *case 1: aperiodic continuous-time signal*
    - *case 2: periodic continuous-time signal*
    - *case 3: aperiodic discrete-time signal*
    - *case 4: periodic discrete-time signal*
  - *The discrete Fourier series*
  - *The sampling of the Fourier transform*
  - *The discrete Fourier transform (DFT)*
    - *definition*
    - *Properties of the DFT*
      - *linearity*
      - *circular shift*
      - *duality*
      - *symmetry*
      - *circular convolution*

The Discrete Fourier Transform (DFT)

- Concept
  - the different faces of the Fourier synthesis/analysis…

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{jk\frac{2\pi}{N}n}$$

The discrete Fourier transform
(N-periodic in n and k)

$$x[n]=\frac{1}{2\pi}\int_{\omega=-\pi}^{\pi}\tilde{X}\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

The discrete-time Fourier transform
(2π-periodic in ω)

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(\Omega)e^{j\Omega t}d\Omega$$

The continuous-time Fourier transform
(in t and Ω)

$$\tilde{x}(t)=\frac{1}{2}a_0+\sum_{k=1}^{+\infty}\left[a_k\cos\left(k\frac{2\pi}{T}t\right)+b_k\sin\left(k\frac{2\pi}{T}t\right)\right]$$

$$\tilde{x}(t)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}c_k\,e^{jk\frac{2\pi}{T}t}$$

The Fourier series
(T-periodic in t)

The Discrete Fourier Transform (DFT)

- Concept

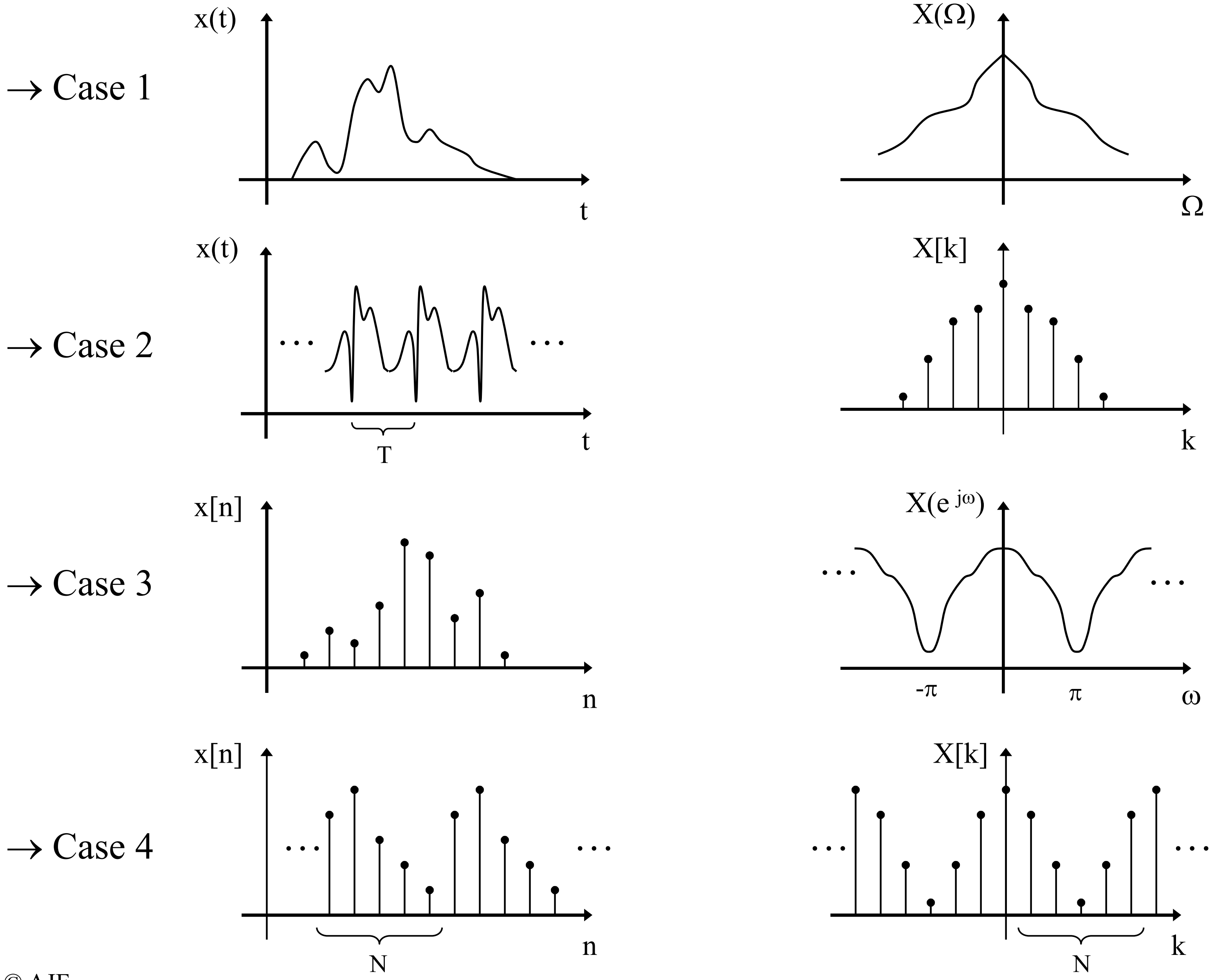
  - It is an alternative to the Fourier transform or to the Z transform to represent <u>finite sequences</u> describing discrete-time signals and linear time-invariant systems,

  - The DFT is a discrete sequence, while the Fourier transform or the Z transform are functions of continuous variables,

  - the DFT corresponds to a sampling of the Fourier transform using equidistant samples in frequency,

  - the DFT is very important in many signal processing applications because efficient algorithms exist (*e.g.*, the FFT, as we shall see) allowing the fast computation of the DFT, which permits the utilization of the DFT, for example, in real-time spectral analysis applications.

The Fourier representation of signals



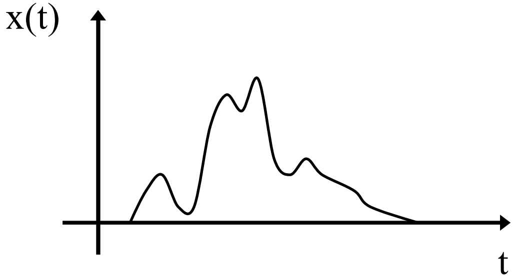
The Fourier representation of signals

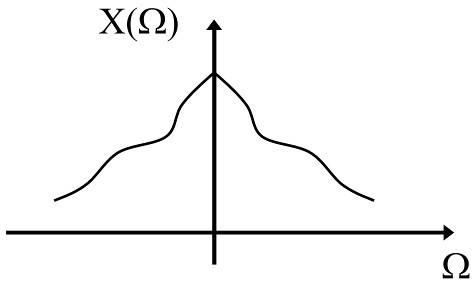
- Review on the Fourier representation of signals

  – we should be familiar already with the Fourier representation of aperiodic continuous-time signals, periodic continuous-time signals, and aperiodic discrete-time signals. The Fourier representation of periodic discrete-time signals is another important case of Fourier representation that consists in the discrete Fourier transform.

→ Case 1: aperiodic continuous-time signal

- x(t) is aperiodic, X(Ω) is aperiodic.





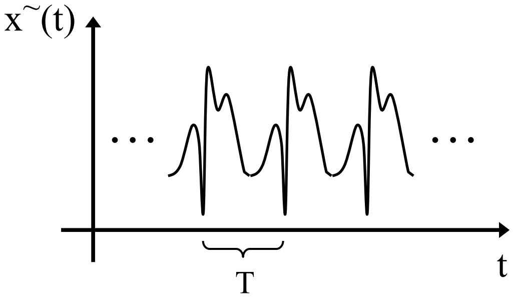
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(\Omega) e^{j\Omega t} d\Omega$$

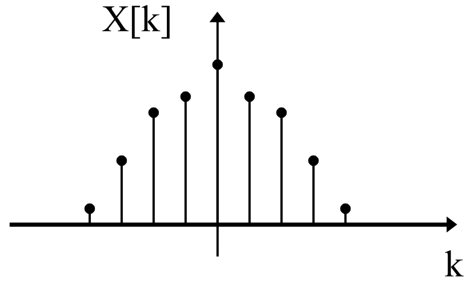
$$X(\Omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\Omega t} dt$$

The Fourier representation of signals

→ Case 2: periodic continuous-time signal

- x~(t) is continuous and periodic (with period T),
- its spectrum, X[k], is described by an aperiodic Fourier series, with an infinite number of coefficients that are associated with complex exponentials whose frequencies are multiple integers (*i.e.,* harmonic) of the fundamental frequency $\Omega=2\pi/T$.





$$\widetilde{x}(t)=\frac{1}{T}\sum_{k=-\infty}^{+\infty}X[k]e^{jk\frac{2\pi}{T}t}$$

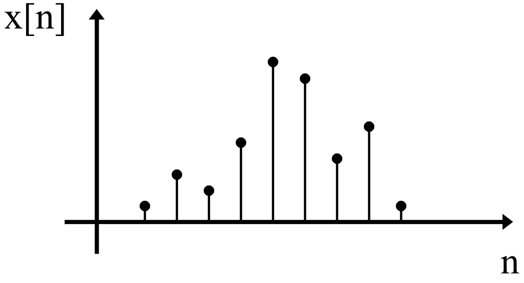
$$X[k]=\int_{T}\widetilde{x}(t)e^{-jk\frac{2\pi}{T}t}dt$$

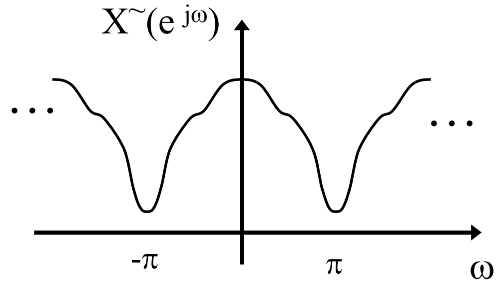
$$\widetilde{x}(t)=\widetilde{x}(t+\ell T) \quad , \quad \forall \ell \quad \text{integer}$$

The Fourier representation of signals

→ Case 3: aperiodic discrete-time signal

- x[n] is aperiodic discrete,
- $X^{\sim}(e^{j\omega})$ is continuous and periodic (with period 2π).





$$x[n] = \frac{1}{2\pi} \int_{2\pi} \widetilde{X}\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

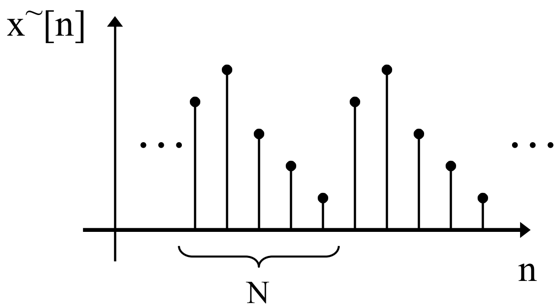
$$\widetilde{X}\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

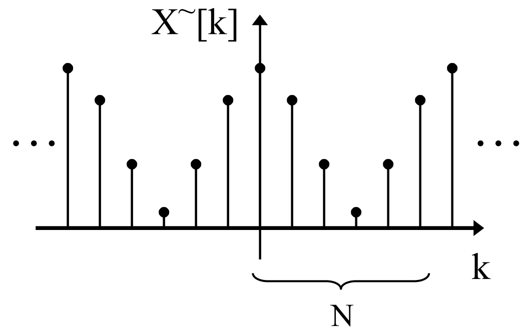
$$\widetilde{X}\left(e^{j\omega}\right) = \widetilde{X}\left(e^{j(\omega+\ell 2\pi)}\right) \quad , \quad \forall \ell \quad \text{integer}$$

The Fourier representation of signals

→ Case 4: periodic discrete-time signal

- x˜[n] is discrete and periodic (with period N),
- the spectrum of X˜[k] is described by an N-periodic Fourier series (N is also the period of the periodic sequence x˜[n]) and their coefficients, X˜[k], are associated with complex exponentials whose frequencies are harmonic of the fundamental frequency ω=2π/N.





$$\tilde{x}[n] = \frac{1}{N}\sum_{N} \tilde{X}[k] e^{jk\frac{2\pi}{N}n}$$

$$\tilde{X}[k] = \sum_{N} \tilde{x}[k] e^{-jk\frac{2\pi}{N}n}$$

$$\tilde{x}[n] = \tilde{x}[n + \ell N] \ , \forall \ \ell \in \mathbb{Z}$$

$$\tilde{X}[k] = \tilde{X}[k + \ell N] \quad , \quad \forall \ell \quad \text{integer}$$

- as a summary…
  - a simple conclusion can be extracted from the four different cases:
    - if the signal is periodic in one domain [time (t or n) or frequency (ω or k)], the signal consists in a set of "lines" in the other domain (frequency or time),
  - the fourth case (periodic Fourier series) is particularly interesting because:
    - it verifies in both domains the two conditions of periodicity and representation using "lines",
    - only N points are necessary in the discrete n domain, or in the discrete frequency domain K, to describe completely a period of the signal.

The discrete Fourier series

- definition
  - consists in the following Fourier pair that uses N points involving one period of the representation in n, or N points involving one period of the discrete representation in the frequency domain (the tilde symbolizes periodicity):

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \tilde{X}[k]=\sum_{n=0}^{N-1}\tilde{x}[n]W_N^{kn}$$

where: $W_N=e^{-j\frac{2\pi}{N}}$

  - Example: given a periodic signal with period N: $\tilde{x}[n]=\sum_{r=-\infty}^{+\infty}\delta[n-rN]=\begin{cases}1, & n=rN, \quad r \text{ integer}\\ 0, & \text{other}\end{cases}$

    and given that in a period only one non-zero impulse exists, we have:

$$\tilde{X}[k]=\sum_{n=0}^{N-1}\delta[n]W_N^{kn}=W_N^0=1 \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}W_N^{-kn}=\frac{1}{N}\sum_{k=0}^{N-1}e^{j\frac{2\pi}{N}kn}$$

but since: $\sum_{k=0}^{N-1}e^{j\frac{2\pi}{N}kn}=\begin{cases}N, & \text{n multiple of N} \quad \Leftrightarrow n=\ell N\\ 0, & \textit{other}\end{cases}$

finally: $\tilde{x}[n]=\sum_{\ell=-\infty}^{+\infty}\delta[n-\ell N]$

NOTE: this result is important to characterize the concept of decimation.

- Sampling of the n-discrete Fourier transform
  - there is a *very* important relation between the Fourier series of a periodic discrete signal (in n) with period N, and the Fourier transform of an aperiodic discrete signal whose length is N:
    - sampling the Fourier transform of a discrete-time signal with length N, using N points uniformly distributed (with spacing $2\pi/N$) in the frequency between 0 and 2π, is equivalent to make x[n] periodic with period N.
  - Example:

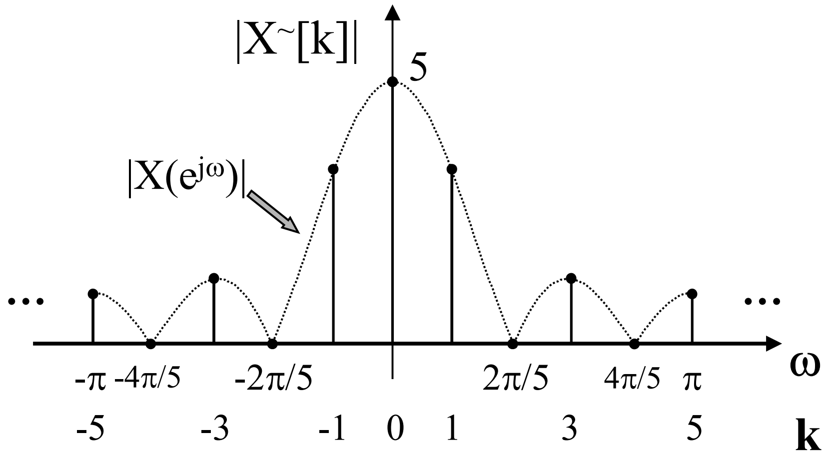
    Represent the Fourier transform of x[n]=1, $0 \leq n \leq 4$, and obtain the sequence $\tilde{x}[n]$ that results from sampling $X(e^{j\omega})$ uniformly in frequency using 10 points: $k2\pi/10$, $0 \leq n \leq 9$.

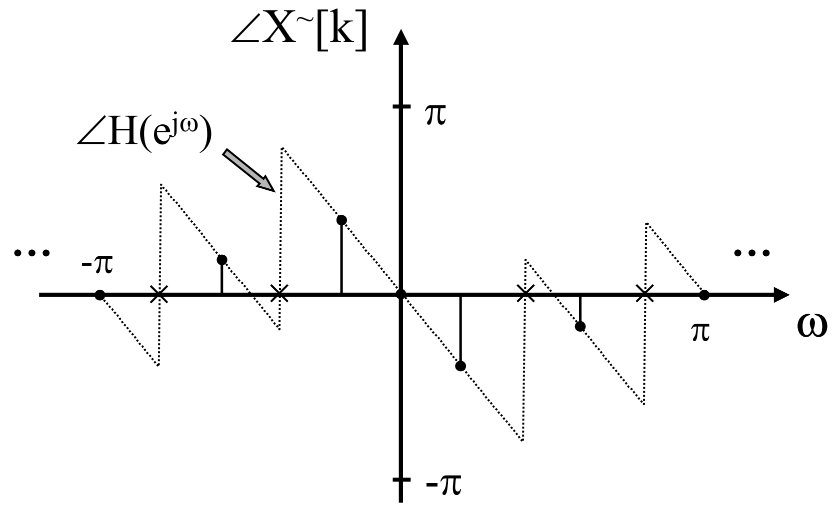
A:

$$x[n] = \begin{cases} 1, & 0 \leq n \leq 4 \\ 0, & other \end{cases} \quad \overset{F}{\longleftrightarrow} \quad X\left(e^{j\omega}\right) = e^{-j2\omega} \cdot \frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

The sampling of the Fourier transform

sampling the Fourier transform we have:

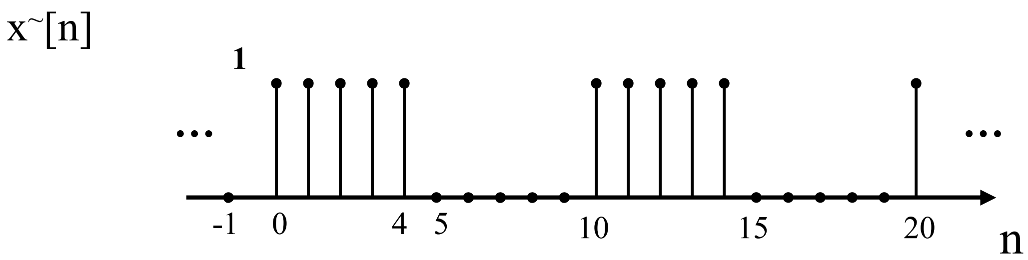




Note: the symbol × in the phase representation means an undefined value since the magnitude is zero.

The sampling of the Fourier transform leads to:

$$\tilde{x}[n] = x[n] * \sum_{\ell=-\infty}^{+\infty} \delta[n-10\ell] = \sum_{\ell=-\infty}^{+\infty} x[n-10\ell]$$



The sampling of the Fourier transform

– the result of the previous example may be presented in a more formal way. If x[n] is an aperiodic sequence having Fourier transform $X(e^{j\omega})$, its sampling for $\omega=k2\pi/N$:

$$\tilde{X}[k]=X\left(e^{j\omega}\right)\Big|_{\omega=k\frac{2\pi}{N}}=X\left(e^{j\frac{2\pi}{N}k}\right)$$

gives rise to a sequence $X^{\sim}[k]$ that is periodic in k, with period N, that may alternatively be obtained using:

$$\tilde{X}[k]=X(\mathrm{Z})\Big|_{\mathrm{Z}=e^{j\frac{2\pi}{N}k}}=X\left(e^{j\frac{2\pi}{N}k}\right)$$

The sequence $X^{\sim}[k]$ may be seen as the Fourier series of a periodic signal $x^{\sim}[n]$ which may be synthesized using a single period of $X^{\sim}[k]$:

$$\tilde{x}[n]=\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]W_N^{-kn}$$

but since:

$$X\left(e^{j\omega}\right)=\sum_{m=-\infty}^{+\infty}x[m]e^{-j\omega m}$$

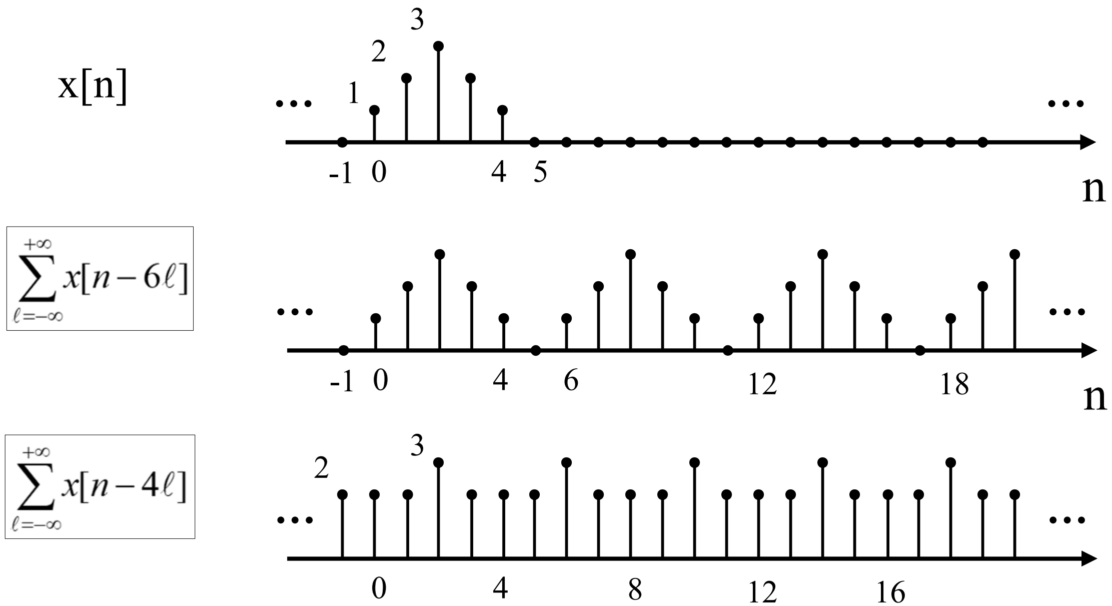
The sampling of the Fourier transform

then:
$$\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{m=-\infty}^{+\infty} x[m]\, e^{-j\frac{2\pi}{N}km}\right] W_N^{-kn} = \sum_{m=-\infty}^{+\infty} x[m]\left[\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right] = \sum_{m=-\infty}^{+\infty} x[m]\,\tilde{p}[n-m]$$

where:
$$\tilde{p}[n-m] = \frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)} = \frac{1}{N}\sum_{k=0}^{N-1} e^{jk\frac{2\pi}{N}(n-m)} = \begin{cases} 1, & n-m=\ell N \\ 0, & other \end{cases} \Leftrightarrow \sum_{\ell=-\infty}^{+\infty} \delta[n-m-\ell N]$$

and finally:
$$\tilde{x}[n] = x[n] * \tilde{p}[n] = x[n] * \sum_{\ell=-\infty}^{+\infty} \delta[n-\ell N] = \sum_{\ell=-\infty}^{+\infty} x[n-\ell N]$$

→ we conclude then that <u>sampling the Fourier transform</u> of an aperiodic signal x[n], using N points uniformly distributed in [0, 2π[, <u>gives rise to the superposition</u> of an infinite number of <u>shifted replicas of x[n]</u>. There is however the risk that the superposition in n ("*aliasing* in time") prevents x[n] from being recognized in the periodic sequence, as the following example illustrates:



The sampling of the Fourier transform

- as a summary…
  - the important conclusion that can be extracted from the previous is that in order to recover x[n] from the periodic sequence $\tilde{x}[n]$, it is necessary that the sampling of $X(e^{j\omega})$ be performed using a <u>number of points N that is equal or greater than the length of </u>x[n].

  - if this condition is satisfied, it is possible to recover x[n] from $\tilde{x}[n]$:

$$x[n]=\begin{cases}\tilde{x}[n], & 0\le n\le N-1\\ 0, & other\end{cases}$$

  - This discussion is reminiscent of the discussion relative to the uniform sampling of continuous signals:
    - taking a band-limited continuous signal $x_c(t)$, there is no loss of information if instead of being represented for all **t** (continuous), the signal is represented by the samples $x[n]=x_c(nT)$ taken uniformly in time,
  - in similar terms, we may also conclude that:
    - taking a finite length x[n] signal, there is no loss of information if instead of being represented for all ω (continuous), $X(e^{j\omega})$ is represented by **N** uniformly distributed samples in frequency, where N is equal or larger than the length of x[n]. This is the concept underlying the Discrete Fourier Transform (DFT) .

The Discrete Fourier Transform (DFT)

- Definition
  - consists in the representation of a finite-length discrete sequence, with x[n]≠0, for $0 \le n \le N-1$, by N values of x[n] or, equivalently, by N values of its frequency-domain representation X[k], on the basis of the implicit assumption that this discrete frequency representation corresponds, in fact, to the description of a periodic signal, one period of which corresponds to x[n].

→ analysis equation of the DFT: $$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}$$ where: $$W_N = e^{-j\frac{2\pi}{N}}$$

→ synthesis equation of the DFT: $$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$$

  - this perspective is of great practical interest (why ?) but we should not forget that in reality and implicitly, we deal with x˜[n] and with X˜[k], and that we only consider (in order to simplify):

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \le n \le N-1 \\ 0, & other \quad n \end{cases}$$

$$X[k] = \begin{cases} \tilde{X}[k], & 0 \le k \le N-1 \\ 0, & other \quad k \end{cases}$$

as a summary: periodicity is intrinsic to the definition of the DFT, which naturally constrains its properties.

- **Example**: to compute the DFT sequence of length N of the following signal:

$$x[n] = \cos\left(n\ell \frac{2\pi}{N}\right), \quad 0 \le n, \ell \le N-1$$

A: it is easy to conclude that: $x[n] = \frac{1}{2}\left(e^{-j\frac{2\pi}{N}n\ell} + e^{j\frac{2\pi}{N}n\ell}\right) = \frac{1}{2}\left(W_N^{n\ell} + W_N^{-n\ell}\right)$

and as: $X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = \frac{1}{2}\left[\underbrace{\sum_{n=0}^{N-1} W_N^{(k+\ell)n}} + \sum_{n=0}^{N-1} W_N^{(k-\ell)n}\right]$

its value is N for k+ℓ=rN, with r integer, but since 0≤k≤N-1, then there is only one possibility ∴ k=N-ℓ.

it results that: $X[k] = \begin{cases} N/2, & k = \ell \\ N/2, & k = N-\ell \\ 0, & k \in [0, N-1] \setminus \{\ell, N-\ell\} \end{cases}$

NOTE: in this case, there is an alternative way to get to the same result: using the inverse DFT.

QUESTION: how to interpret the case where $\ell=0$?

Properties of the DFT

- Linearity

$$x_3[n] = ax_1[n] + bx_2[n] \quad \overset{F}{\longleftrightarrow} \quad X_3[k] = aX_1[k] + bX_2[k]$$

length of $x_1[n] \rightarrow N_1$

length of $x_2[n] \rightarrow N_2$

$\therefore$ length of $x_3[n] \rightarrow N_3 = MAX(N_1, N_2)$

NOTE: the shortest sequence must be extended by appending zeroes (a process that is known as “zero-padding”) till it matches the length of the longer sequence, previously to the computation of the DFTs.

- Circular time shift

considering: $x[n] \overset{F}{\longleftrightarrow} X[k]$

we want to know:

$$x_1[n] = ? \quad \overset{F}{\longleftrightarrow} \quad e^{-j\frac{2\pi}{N}km} X[k]$$

Properties of the DFT

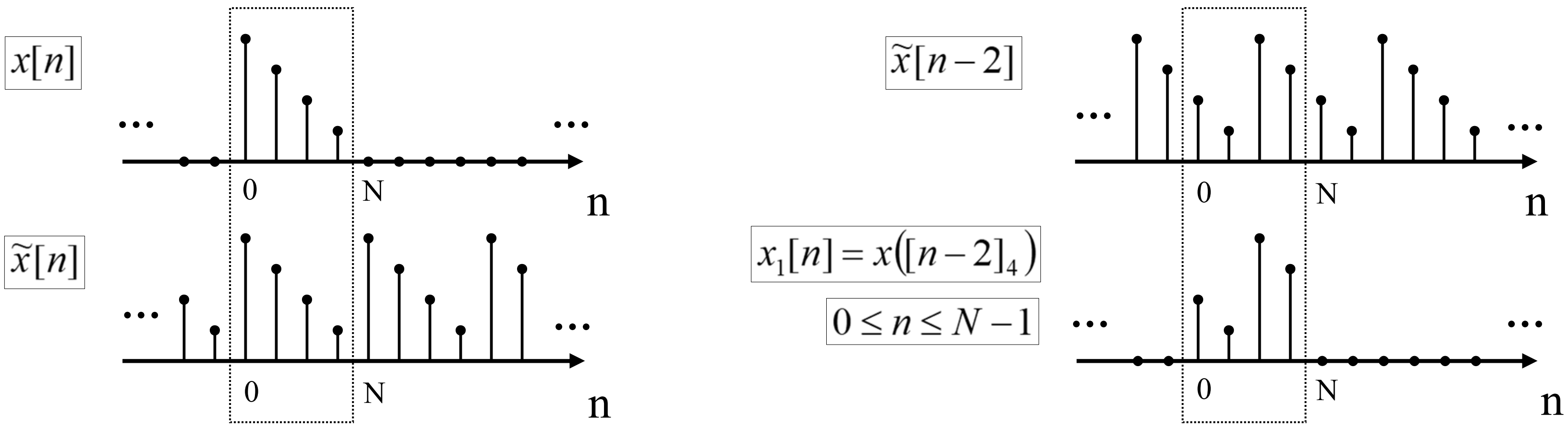
if: $\widetilde{x}[n]=\sum_{\ell=-\infty}^{+\infty} x[n-\ell N]=x([n \text{ modulo } N])=x([n]_N)$

and as we know that:

$$\widetilde{x}_1[n]=\widetilde{x}[n-m] \quad \overset{F}{\longleftrightarrow} \quad \widetilde{X}_1[k]=e^{-j\frac{2\pi}{N}km}\widetilde{X}[k]$$

we conclude that: $x_1[n]=\begin{cases}\widetilde{x}_1[n]=x([n-m]_N), & 0\le n\le N-1\\ 0, & other \quad n\end{cases}$

where $x([n-m]_N)$ represents the circular shift of x[n] as illustrated in the following example where N=4 and m=2:



NOTE 1: given the nature of the circular shift, then: $x([n-\ell]_N)=x([n+(N-\ell)]_N)$

since: $W_N^{k\ell}=W_N^{-k(N-\ell)}$

Properties of the DFT

NOTE 2: using a similar procedure, it can also be concluded that :

$$e^{j\frac{2\pi}{N}n\ell}x[n] \xleftrightarrow{F} X([k-\ell]_N)$$

- Duality

if:

$$\tilde{x}[n] \xleftrightarrow{F} \tilde{X}[k]$$

it results, considering the duality property of the Fourier series:

$$\tilde{X}[n] \xleftrightarrow{F} N\,\tilde{x}[-k]$$

and, therefore, if:

$$x[n] \xleftrightarrow{F} X[k]$$

it results also that:

$$X[n] \xleftrightarrow{F} N\,x([-k]_N), \quad 0 \le k \le N-1$$

Properties of the DFT

- Symmetry

defining the following N-length sequences:

$\rightarrow$ periodic conjugate-symmetric component:

$$x_{ep}[n] = \tilde{x}_e[n] = \frac{1}{2}\left(\tilde{x}[n] + \tilde{x}^*[-n]\right) = \frac{1}{2}\left(x[n] + x^*[N-n]\right), \quad 0 \le n \le N-1$$

$\rightarrow$ periodic conjugate-antisymmetric component:

$$x_{op}[n] = \tilde{x}_o[n] = \frac{1}{2}\left(\tilde{x}[n] - \tilde{x}^*[-n]\right) = \frac{1}{2}\left(x[n] - x^*[N-n]\right), \quad 0 \le n \le N-1$$

it results: $x[n] = x_{ep}[n] + x_{op}[n]$

we may also conclude [Oppenheim, section 8.64]:

$$\left.\begin{array}{r} \Re e\{x[n]\} \\ j\Im m\{x[n]\} \\ x_{ep}[n] \\ x_{op}[n] \end{array}\right\} \quad \overset{F}{\longleftrightarrow} \quad \left\{\begin{array}{l} X_{ep}[k] \\ X_{op}[k] \\ \Re e\{X[k]\} \\ j\Im m\{X[k]\} \end{array}\right.$$

Properties of the DFT

NOTE : it is also easy to verify that:

$$\left.\begin{array}{r} x^*[n] \\ x^*([-n]_N) \end{array}\right\} \overset{F}{\longleftrightarrow} \left\{\begin{array}{l} X^*([-k]_N) \\ X^*[k] \end{array}\right.$$

- Circular convolution

If $x_1[n]$ and $x_2[n]$ are two N-length sequences whose DFTs are $X_1[k]$ and $X_2[k]$, respectively, what is $x_3[n]$, the inverse DFT of the product $X_1[k]X_2[k]$?

A: Considering the periodic sequences $\tilde{x}_1[n] = x_1([n]_N)$ and $\tilde{x}_2[n] = x_2([n]_N)$ then:

$$\tilde{x}_3[n] = \tilde{x}_1[n] * \tilde{x}_2[n] = \sum_{\ell=0}^{N-1} \tilde{x}_1[\ell]\tilde{x}_2[n-\ell]$$

which is the periodic convolution.

Properties of the DFT

Using this result it is also :

$$x_3[n]=\sum_{\ell=0}^{N-1} x_1\left([\ell]_N\right)x_2\left([n-\ell]_N\right),\quad 0\le n\le N-1$$

which may be expressed as:

$$x_3[n]=\sum_{\ell=0}^{N-1} x_1[\ell]x_2\left([n-\ell]_N\right)=x_1[n]\otimes x_2[n],\quad 0\le n\le N-1$$

The notation $x_1[n]\otimes x_2[n]$ is representative of the circular convolution because, in its computation, the second sequence is inverted in ℓ and is circularly shifted relative to the length of its period.

NOTE 1: differently from the linear convolution, the result of the circular convolution between two N-length sequences has length N.

NOTE 2: the circular convolution is also commutative and hence:

$$x_1[n]\otimes x_2[n]=x_2[n]\otimes x_1[n]=\sum_{\ell=0}^{N-1} x_2[\ell]x_1\left([n-\ell]_N\right),\quad 0\le n\le N-1$$

Properties of the DFT

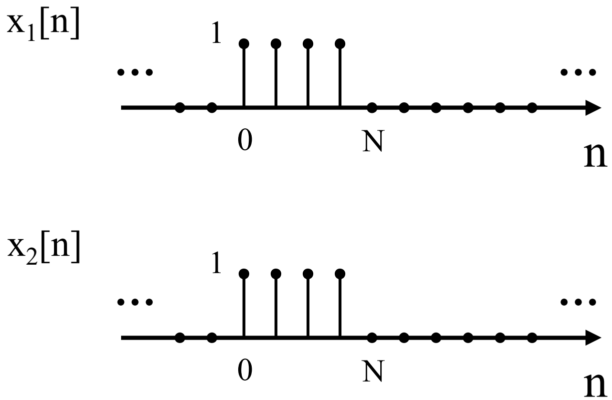
- Example 1:

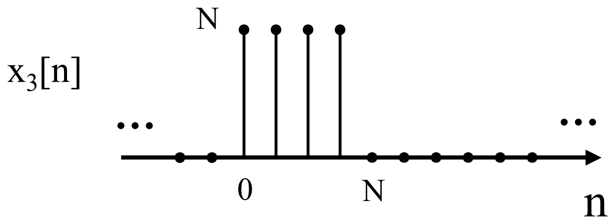
If: $x_1[n] = x_2[n] = \begin{cases} 1, & 0 \le n \le N-1 \\ 0, & outros \quad n \end{cases}$ what is the result of $x_1[n] \otimes x_2[n]$?

A: as: $X_1[k] = X_2[k] = \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N, & k=0 \\ 0, & 1 \le k \le N-1 \end{cases}$

then: $X_3[k] = X_1[k] X_2[k] = \begin{cases} N^2, & k=0 \\ 0, & 1 \le k \le N-1 \end{cases} \quad \overset{F}{\longleftrightarrow} \quad x_3[n] = N, \quad 0 \le n \le N-1$

graphically we have (*e.g.,* N=4):





Properties of the DFT

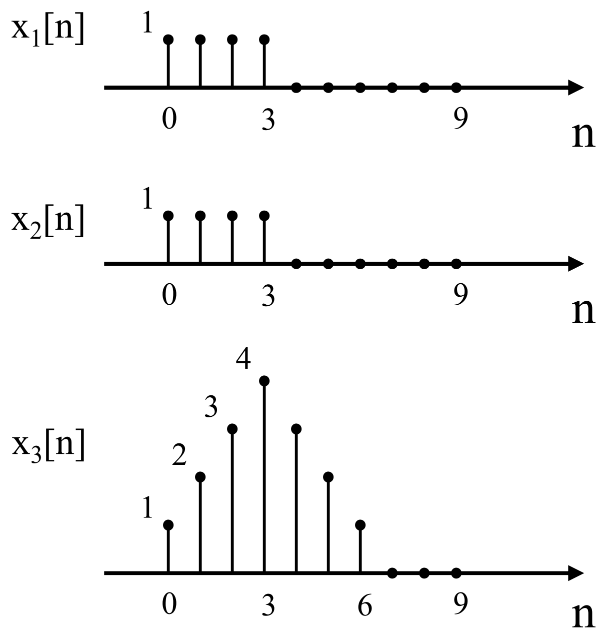
- Example 2:

If: $x_1[n] = x_2[n] = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & L \le n \le N-1 \cup n < 0 \cup n > N \end{cases}$ what is the result of $x_1[n] \otimes x_2[n]$?

A: as: $X_1[k] = X_2[k] = \sum_{n=0}^{L-1} W_N^{kn} = \dfrac{1-W_N^{kL}}{1-W_N^{k}}$

then: $X_3[k] = X_1[k] X_2[k] = \left(\dfrac{1-W_N^{kL}}{1-W_N^{k}} \right)^2$

admitting N=10 and L=4, graphically we have:



Question 1: May we state that in this example the result of the circular convolution is the same as that of the linear convolution ?

Question 2: Keeping L=4, what is the minimum value of N that leads to the same result ?

Question 3: May we state that we may use the circular convolution to compute the linear convolution ? If yes, under which conditions ?

Properties of the DFT

– It can also be shown that:

$$x_1[n]\cdot x_2[n] \quad \overset{F}{\longleftrightarrow} \quad \frac{1}{N}X_1[k]\otimes X_2[k]=\frac{1}{N}\sum_{\ell=0}^{N-1}X_1[\ell]X_2\left([k-\ell]_N\right)$$