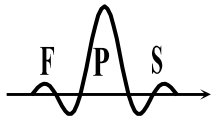


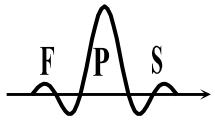
Summary

- *Sampling and reconstruction of continuous signals*
 - *Introduction*
 - *Periodic sampling of continuous-time signals*
 - *Frequency domain analysis of periodic sampling*
 - *Reconstruction of continuous-time signals from samples*
 - *Ideal reconstruction*
 - *Zero-order real reconstruction*
 - *Discrete-time processing of continuous-time signals*



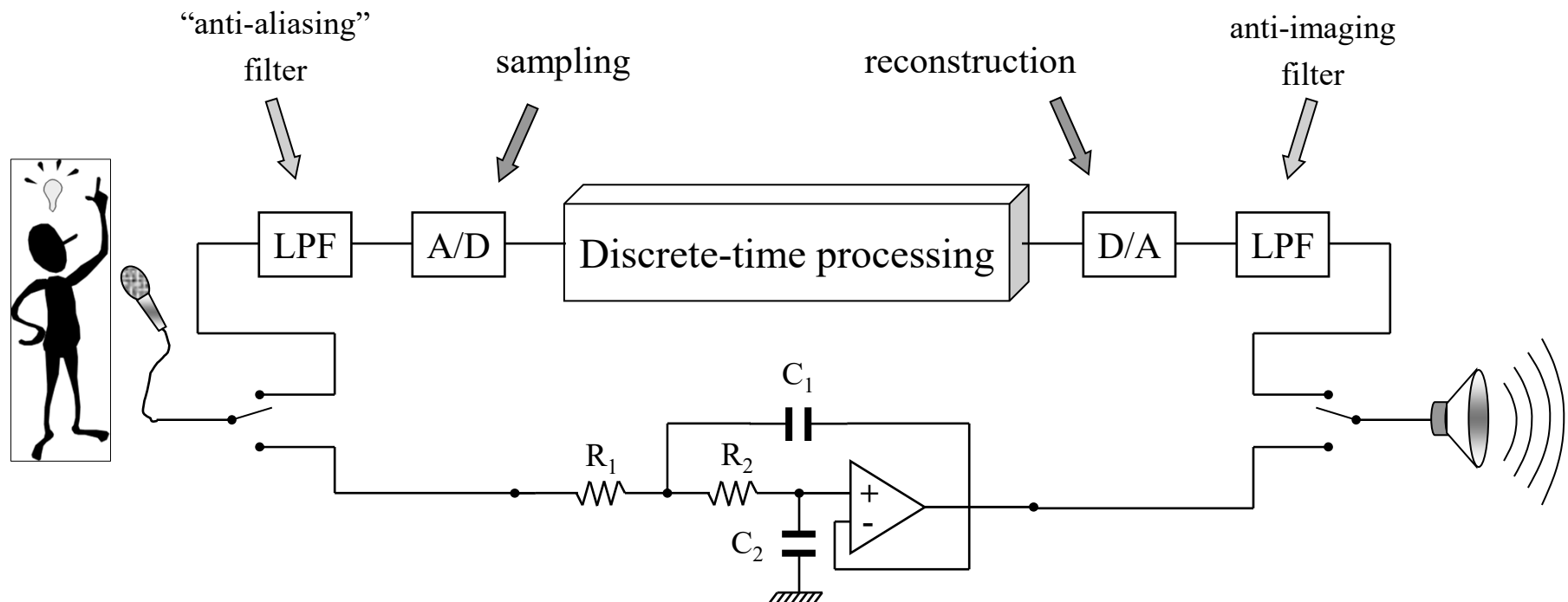
Sampling of continuous-time signals

- Introduction
 - most discrete-time signals result from sampling (*i.e.* discretization in time) of continuous-time signals
 - under certain conditions, a discrete-time signal may be an exact representation (*i.e.* there is no loss of information) of a continuous-time signal
 - any form of processing of a continuous-time signal may be realized in the discrete domain, which requires the sampling of the continuous-time signal before processing, and the reconstruction of the continuous-time signal from samples after the processing stage

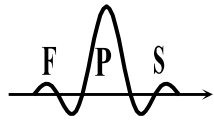


Sampling of continuous-time signals

- Introduction (cont.)
 - is discrete-time processing preferable to analog processing ?

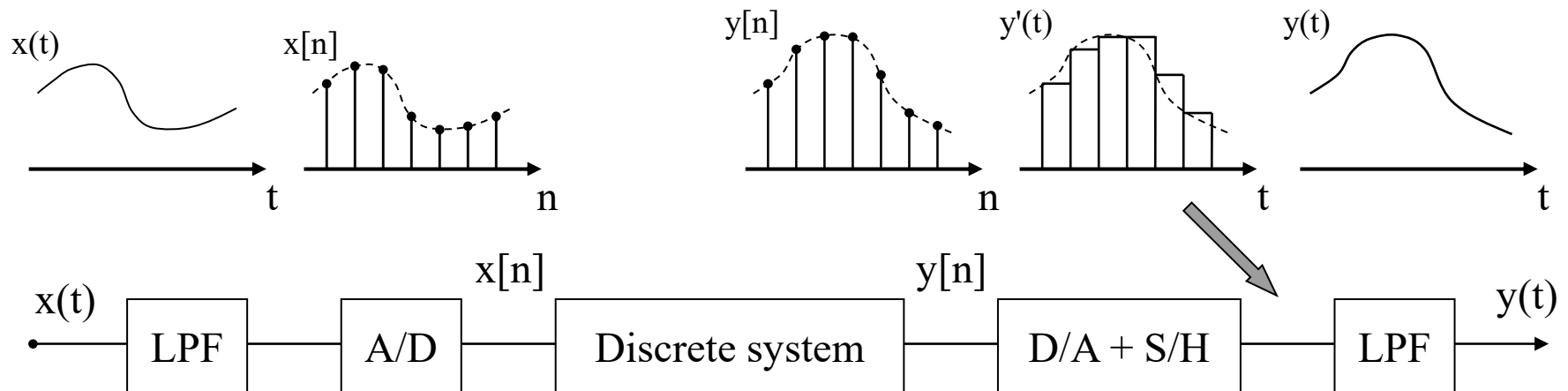


- e.g. is there any non-trivial analog filter with exact linear phase ? (but easy to realize using a discrete-time system...)

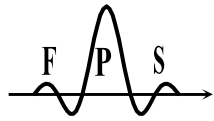


Sampling of continuous-time signals

- Context
 - minimal structure for the discrete-time processing of analog signals:



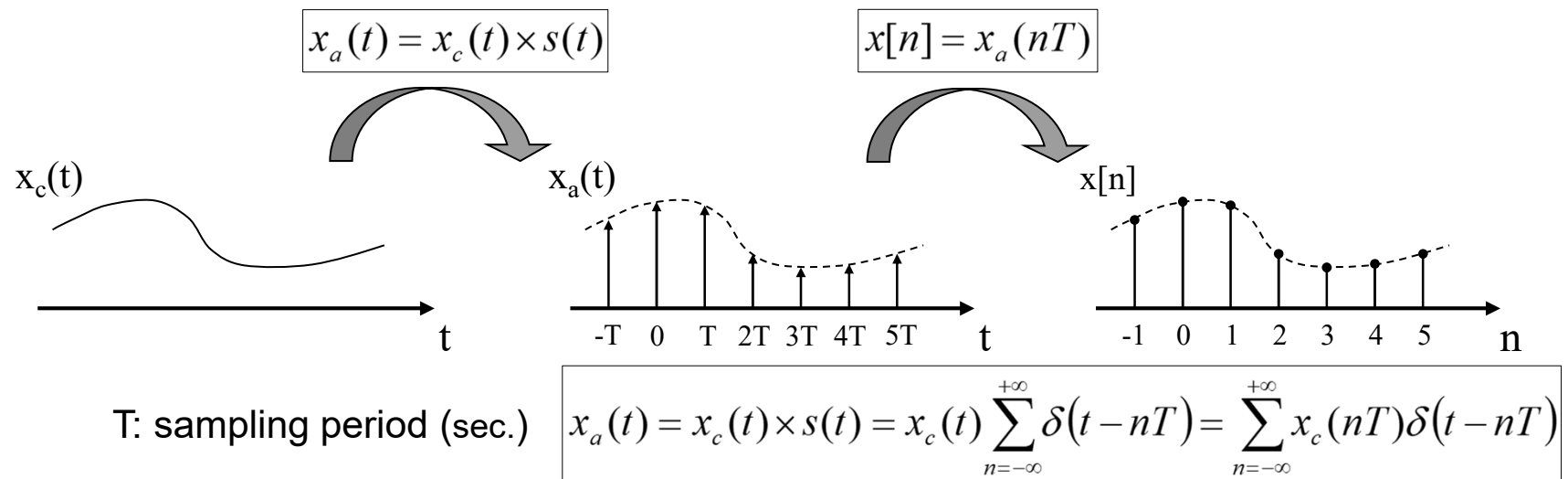
- in the following we admit that the sampling rate is constant and that the A/D and D/A converters have infinite resolution (i.e., no quantization errors)
- QUESTION: in the absence of discrete-time processing, i.e., if $y[n]=x[n]$, and admitting ideal A/D and D/A converters, under which conditions is it possible to sample and reconstruct an analog signal without loss of information, i.e., such that $y(t)=x(t)$?



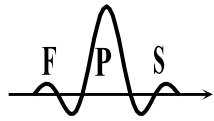
Sampling of continuous-time signals

- in order to answer the previous question, we analyze two fundamental steps in the represented block diagram : the time discretization of the continuous-time signal by means of a periodic sampling (continuous-time signal $\rightarrow$ discrete-time signal conversion) and the time reconstruction of the continuous-time signal from samples (discrete-time signal $\rightarrow$ continuous-time signal conversion)

- periodic sampling

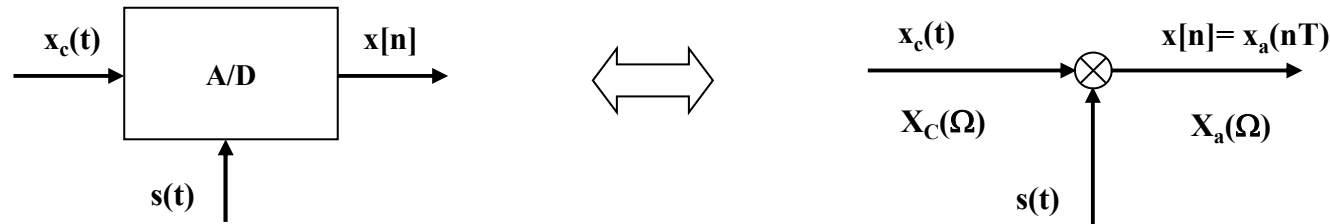


- NOTE: this operation is only invertible (*i.e.*, the ambiguity is avoided of two different signals giving rise to the same discrete signal) if $x_c(t)$ is constrained.



Frequency domain analysis of periodic sampling

- time discretization: how to relate $X(e^{j\omega})$ and $X_c(\Omega)$?



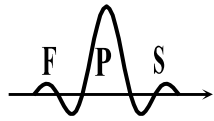
$$\left\{ \begin{array}{l} s(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT) \\ \dots \uparrow \uparrow \uparrow \uparrow \uparrow \dots \\ -2T \quad -T \quad 0 \quad T \quad 2T \quad t \end{array} \right\} \xleftrightarrow{F} \left\{ \begin{array}{l} S(\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta(\Omega - k \frac{2\pi}{T}) \\ \dots \uparrow \uparrow \uparrow \uparrow \uparrow \dots \\ -4\pi/T \quad -2\pi/T \quad 0 \quad 2\pi/T \quad 4\pi/T \quad \Omega \end{array} \right.$$

$$x_a(t) = x_c(t) \cdot s(t) = \sum_{n=-\infty}^{+\infty} x_c(nT) \delta(t - nT) \xleftrightarrow{F} X_a(\Omega) = \frac{1}{2\pi} X_c(\Omega) * S(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c(\Omega - k \frac{2\pi}{T})$$

and also:

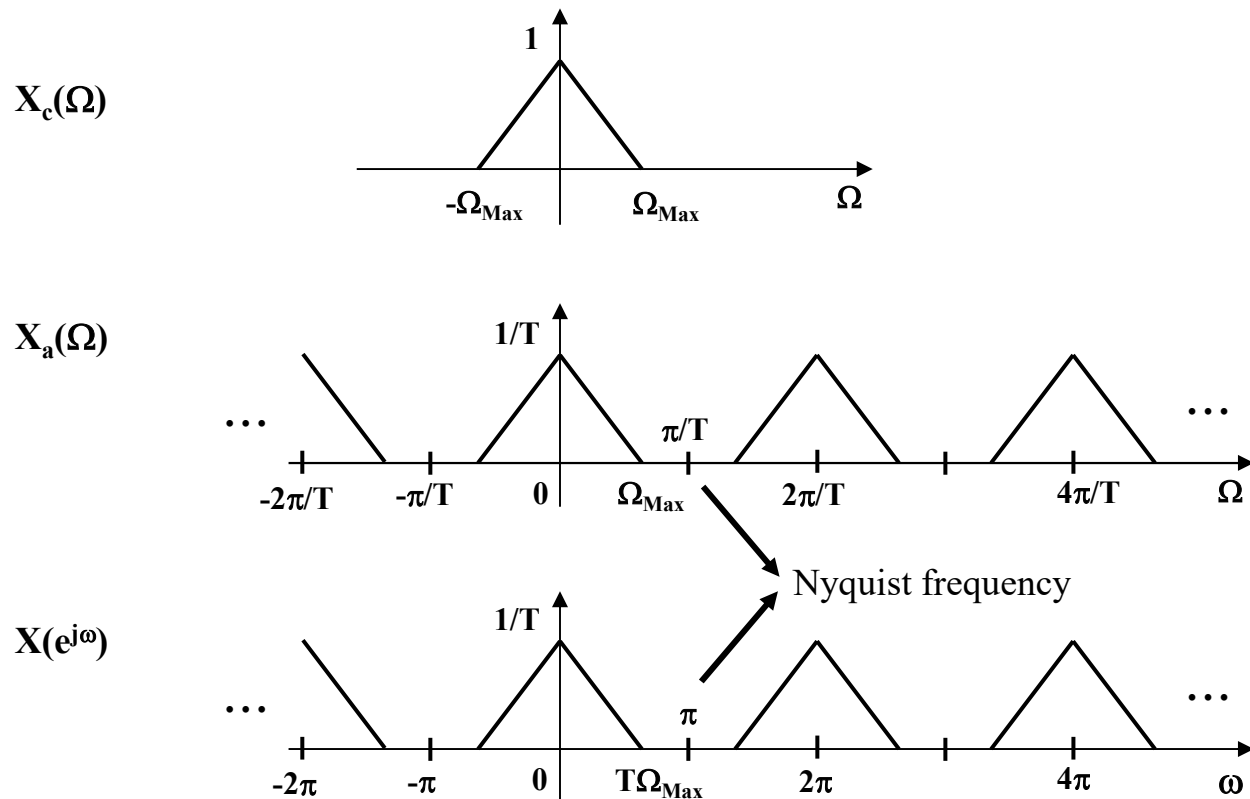
$$x_a(t) = \sum_{n=-\infty}^{+\infty} x_c(nT) \delta(t - nT) \xleftrightarrow{F} X_a(\Omega) = \sum_{n=-\infty}^{+\infty} x_c(nT) e^{-jn\Omega T} \Big|_{\substack{x[n]=x_c(nT) \\ \omega=\Omega T}} = \sum_{n=-\infty}^{+\infty} x[n] e^{-jn\omega} = X(e^{j\omega})$$

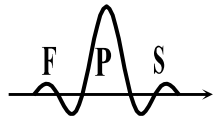
thus:
$$X(e^{j\omega}) = X_a(\Omega) \Big|_{\Omega=\frac{\omega}{T}} = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c\left(\frac{\omega - k2\pi}{T}\right)$$



Frequency domain analysis of periodic sampling

→ The previous result says that except for a scale factor and a normalization (by $1/T$) of the frequency axis (making that the “analog” frequency $k2\pi/T = k\Omega_s$ be projected in the “digital” frequency $k2\pi$, for any integer K) the spectra $X(e^{j\omega})$ and $X_a(\Omega)$ are similar. It also says that, as result of ideal sampling, the spectrum of the continuous-time signal appears replicated at all multiple integers of the sampling frequency.





Frequency domain analysis of periodic sampling

The Nyquist sampling theorem

- in order to avoid spectral overlap (i.e., *aliasing*) between replicas of the base-band spectrum, it must be ensured that :

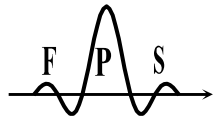
$$\Omega_{\text{MAX}} < \pi/T = \Omega_S/2 \Leftrightarrow 2\pi F_{\text{MAX}} < \pi F_S \Leftrightarrow F_S > 2F_{\text{MAX}}$$

- this means that the bandwidth of the base-band signal must be limited to less than half the sampling frequency. This condition is typically enforced by a low-pass filter just before the A/D converter, thus named “anti-aliasing” filter.
- if this condition is guaranteed, as the illustration suggests, it is possible to recover $X_c(\Omega)$ from $X(e^{j\omega})$, using an ideal low-pass continuous-time filter, with gain T and cut-off frequency $\Omega_{\text{MAX}} < \Omega_p < \Omega_S - \Omega_{\text{MAX}}$

these aspects reflect the Nyquist sampling theorem:

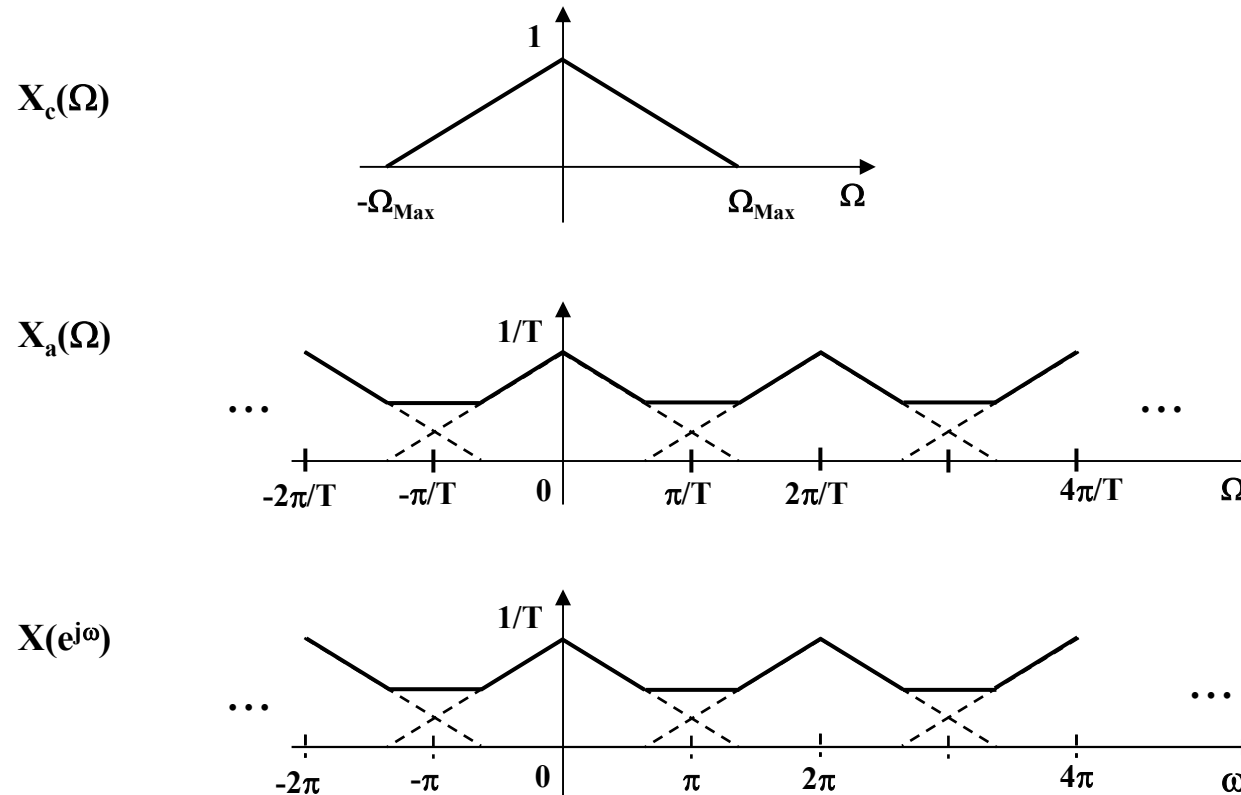
- is $x_c(t)$ is a band-limited signal such that $X_c(\Omega) = 0$ for $|\Omega| > \Omega_{\text{MAX}}$, then $x_c(t)$ is uniquely determined (i.e. may be unambiguously reconstructed) from its samples $x[n] = x_c(nT)$ with $\Omega_S = 2\pi/T > 2\Omega_{\text{MAX}}$

NOTE: $\Omega_S/2 = \pi/T$ is commonly known as the Nyquist frequency.

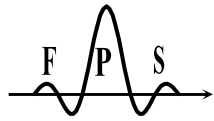


Frequency domain analysis of periodic sampling

→ what if the sampling condition is violated, i.e., if $F_S < 2F_{MAX}$?



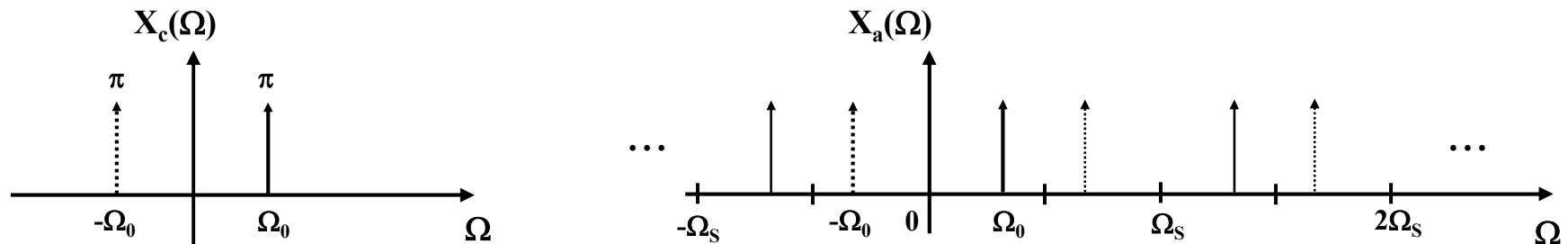
Answer: there is spectral overlap (“aliasing”) distorting the signal, and preventing the recovery of the original spectrum after low-pass filtering.



Frequency domain analysis of periodic sampling

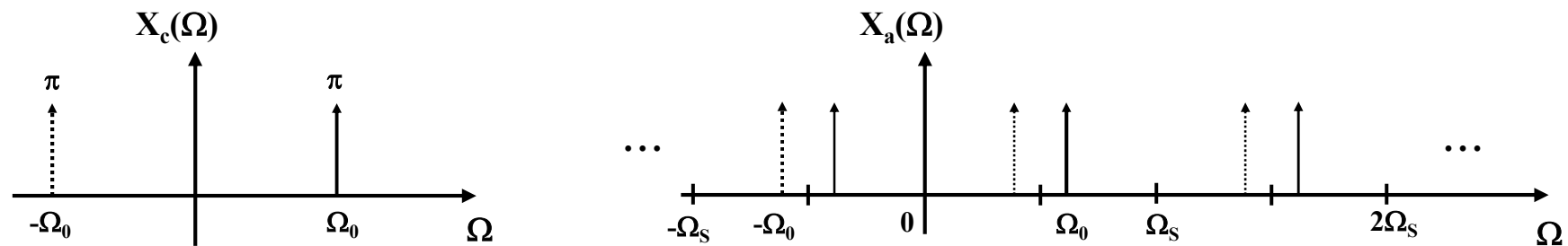
example: case of a continuous-time signal (co-sinusoidal function) correctly and incorrectly sampled

$x_c(t) = \cos(\Omega_0 t)$, $\Omega_0 < \Omega_s/2$ $\therefore$ there is no “aliasing”

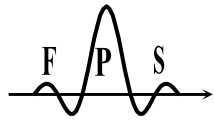


recovered signal after low-pass filtering, with cut-off at $\Omega_s/2$: $x_c(t) = \cos(\Omega_0 t)$

$x_c(t) = \cos(\Omega_0 t)$, $\Omega_0 > \Omega_s/2$ $\therefore$ there is “aliasing”

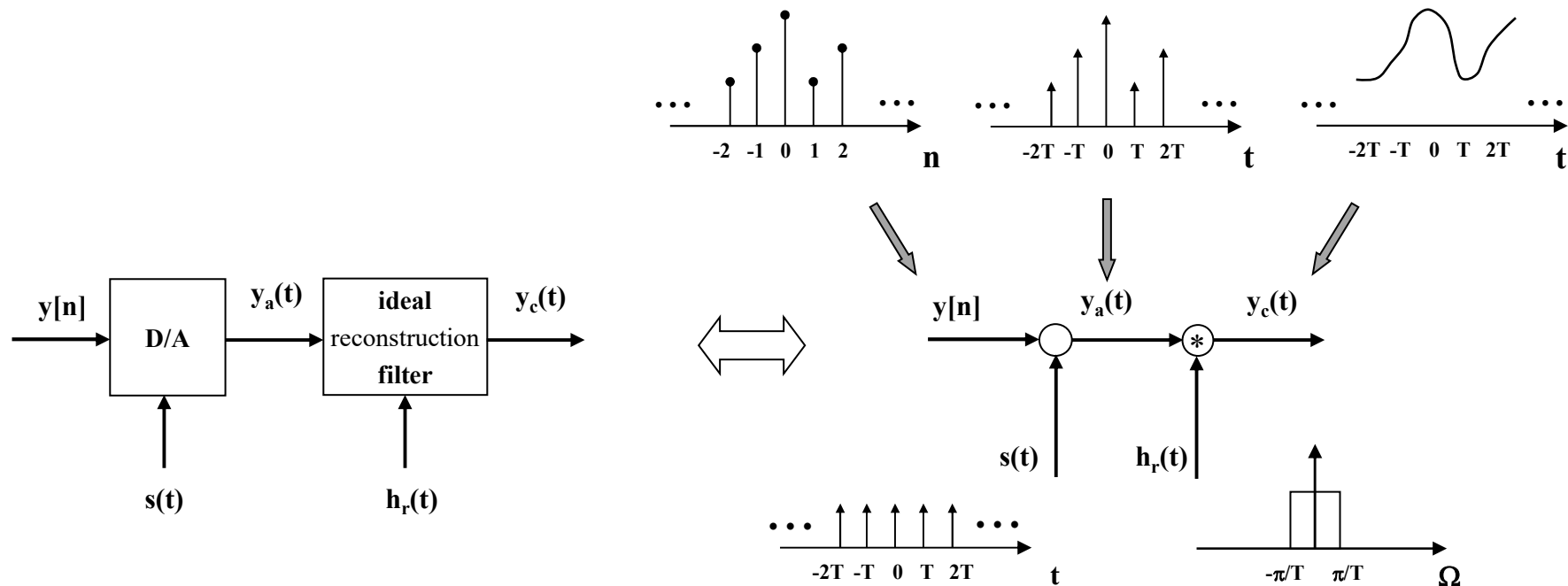


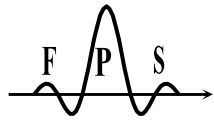
recovered signal after low-pass filtering, with cut-off at $\Omega_s/2$: $x_c(t) = \cos[(\Omega_s - \Omega_0)t]$



Reconstruction from samples

- Case 1: ideal reconstruction
 - as can be concluded from the spectral representation of $X_a(\Omega)$ ('slide' n° 7), if we preserve solely the base-band replica after low-pass filtering, then it is possible to recover the spectrum $X_c(\Omega)$; the same is to say: it is possible to recover $x_c(t)$. This is the principle that we will illustrate next using $y[n]$.





Reconstruction from samples

The first step going from the discrete-time domain to the continuous-time domain involves placing the pulses of the discrete sequence $y[n]$ at instants uniformly distributed in time, thus obtaining $y_a(t)$. It should be noted that this signal has the same spectrum of $x_a(t)$ since we presume that $y[n]=x[n]$.

$$y_a(t) = \sum_{n=-\infty}^{+\infty} y[n] \delta(t - nT) \xleftrightarrow{F} Y_a(\Omega) = \sum_{n=-\infty}^{+\infty} y_a(nT) e^{-jn\Omega T} \Big|_{\substack{y(n)=y_a(nT) \\ \omega=\Omega T}} = \sum_{n=-\infty}^{+\infty} y[n] e^{-jn\Omega T} = Y(e^{j\Omega T})$$

By submitting the continuous-time signal $y_a(t)$ to an ideal low-pass filter having impulse response $h_r(t)$, gain T and cutting-off frequency at π/T :

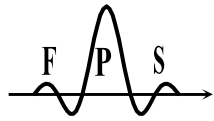
$$H_r(\Omega) = \begin{cases} T, & |\Omega| < \pi/T \\ 0, & \text{other} \end{cases}$$

$\xleftrightarrow{F}$

$$h_r(t) = \frac{\sin \pi t / T}{\pi t / T}$$

we obtain:

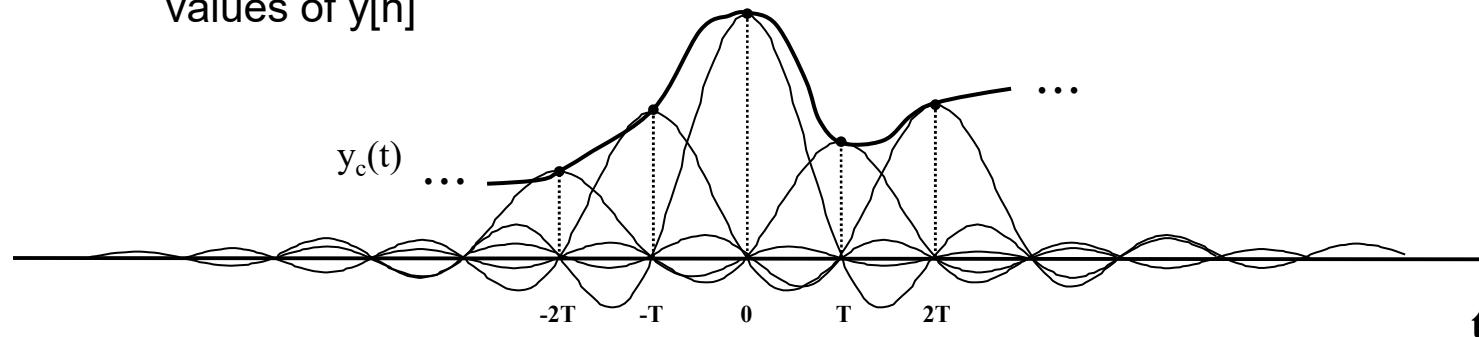
$$y_c(t) = y_a(t) * h_r(t) = \sum_{n=-\infty}^{\infty} y_a(nT) \frac{\sin \pi(t - nT) / T}{\pi(t - nT) / T} = \sum_{n=-\infty}^{\infty} y[n] \text{sinc} \frac{\pi}{T} (t - nT)$$



Reconstruction from samples

This result reveals that:

- at the sampling instants $y_c(nT) = y[n] = x[n] = x_c(nT)$, given that all sinc functions in the summation are zero, except one (that centered at $t = nT$) whose value is 'one',
- at intermediary instants, the continuous-time signal results from the sum of all sinc functions, i.e. the filter $h_r(t)$ implements an interpolation using all values of $y[n]$



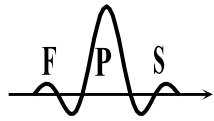
using frequency-domain analysis, and considering $y[n] = x[n]$ which implies:

$$Y_a(\Omega) = Y(e^{j\omega}) \Big|_{\omega=\Omega T} = X(e^{j\omega}) \Big|_{\omega=\Omega T} = X_a(\Omega)$$

It can be concluded that the result of filtering is:

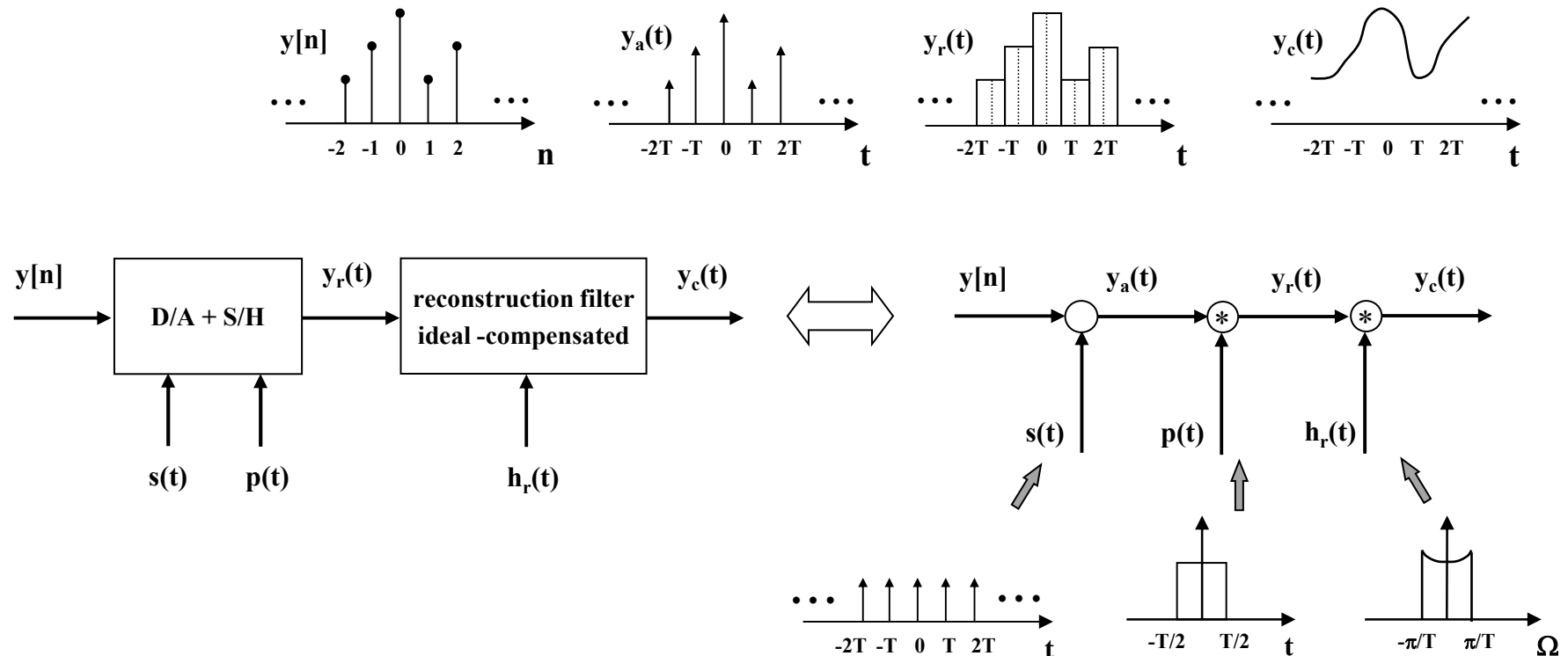
$$y_c(t) = y_a(t) * h_r(t) \quad \xleftrightarrow{F} \quad Y_c(\Omega) = X_a(\Omega) \cdot H_r(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c\left(\Omega - k \frac{2\pi}{T}\right) H_r(\Omega) = X_c(\Omega)$$

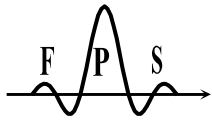
which means that, considering ideal conditions and the Nyquist criterion, it is possible to reconstruct the continuous-time signal from its samples, without loss of information. **Question:** the reconstruction filter is also known as anti-imaging filter, why ?



Reconstruction from samples

- Case 2: zero-order real reconstruction
 - real electronic devices, in particular D/A converters, do not operate using pulses but use instead more physically tractable signals such as boxcar function approximations. Let us consider the case closest to reality where the D/A converter is associated with a “sample-and-hold” device that ‘retains’ the value of a sample during a sampling period, giving rise to a staircase-like signal:





Reconstruction from samples

as considered before:

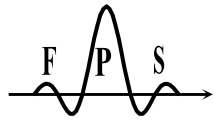
$$y_a(t) = \sum_{n=-\infty}^{+\infty} y[n] \delta(t - nT) \xleftrightarrow{F} Y_a(\Omega) = Y(e^{j\omega}) \Big|_{\omega=\Omega T} = Y(e^{j\Omega T})$$

and for the boxcar function of width T:

$$p(t) = \begin{cases} 1, & |t| < T/2 \\ 0, & \text{outros} \end{cases} \xleftrightarrow{F} P(\Omega) = T \frac{\sin \Omega T / 2}{\Omega T / 2}$$

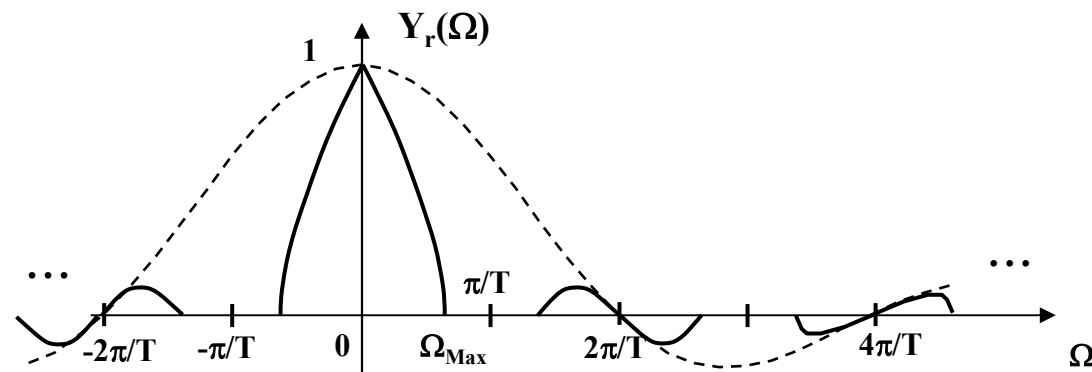
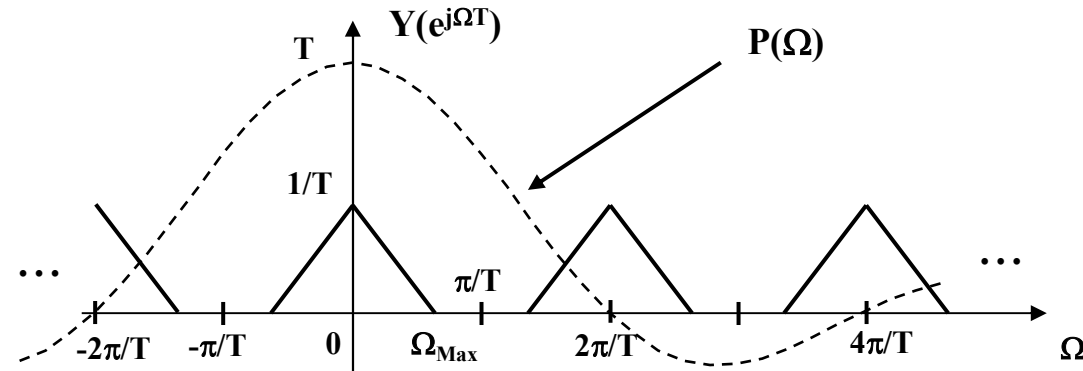
and, therefore, $y_r(t)$ results as:

$$y_r(t) = y_a(t) * p(t) = \sum_{n=-\infty}^{+\infty} y[n] p(t - nT) \xleftrightarrow{F} Y_r(\Omega) = Y(e^{j\Omega T}) T \frac{\sin \Omega T / 2}{\Omega T / 2}$$

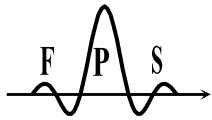


Reconstruction from samples

whose spectral representation is:



from where it can be concluded that the zero-order reconstruction distorts the $Y(e^{j\Omega T})$ spectrum in a way that can be compensated for, if we consider the base-band replica which is the one we want to recover; in addition, all other replicas which we want to eliminate, are strongly attenuated which alleviates the filtering effort of $h_r(t)$.



Reconstruction from samples

The filter $h_r(t)$ must then not only reject the undesirable spectral images, but also compensate the magnitude distortion affecting the base-band replica :

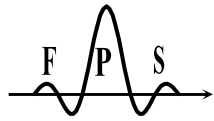
$$y_c(t) = y_r(t) * h_r(t) \quad \xleftrightarrow{F} \quad Y_c(\Omega) = Y_r(\Omega) \cdot H_r(\Omega) = Y(e^{j\Omega T})T \cdot \frac{\sin \Omega T / 2}{\Omega T / 2} \cdot H_r(\Omega)$$

presuming also that $y[n]=x[n]$, then:
$$Y(e^{j\omega}) \Big|_{\omega=\Omega T} = X(e^{j\omega}) \Big|_{\omega=\Omega T} = X_a(\Omega)$$

$$Y_c(\Omega) = X_a(\Omega) \cdot T \frac{\sin \Omega T / 2}{\Omega T / 2} \cdot H_r(\Omega) = \sum_{k=-\infty}^{+\infty} X_c(\Omega - k \frac{2\pi}{T}) \cdot \frac{\sin \Omega T / 2}{\Omega T / 2} \cdot H_r(\Omega) = X_c(\Omega)$$

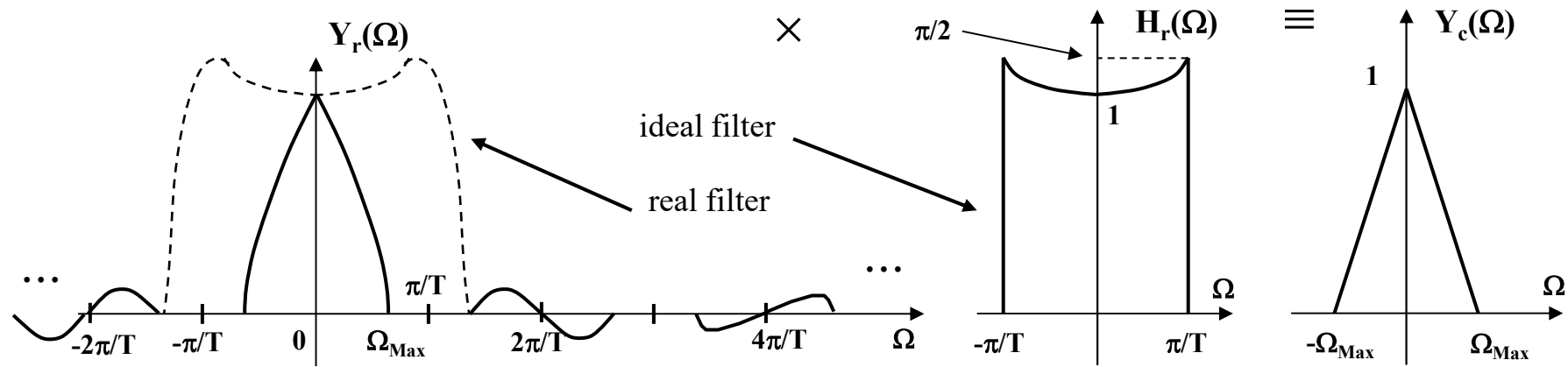
subject to the condition that filter $H_r(\Omega)$ is low-pass, with cut-off frequency at π/T , but is also compensated such as to reverse the $\sin(x)/x$ distortion, i.e. :

$$H_r(\Omega) = \begin{cases} \frac{\Omega T / 2}{\sin \Omega T / 2}, & |\Omega| < \pi / T \\ 0, & \text{other} \end{cases}$$



Reconstruction from samples

Then, it results graphically:

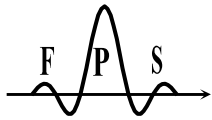


which means the output of $h_r(t)$ is also given by:
as we have already concluded before.

$$y_c(t) = \sum_{n=-\infty}^{\infty} x_c(nT) \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T}$$

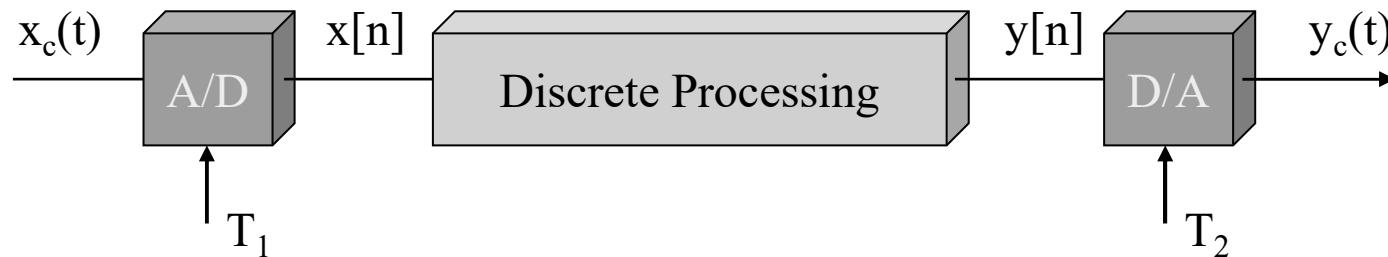
NOTE 1: the compensation $\sin(x)/x$ may be inserted at any stage of the processing, including (and perhaps preferably !) at the discrete processing stage, with all the known advantages.

NOTE 2: in addition to the 'zero-order' reconstruction, there are other possibilities (e.g. the 'one-order' reconstruction) !

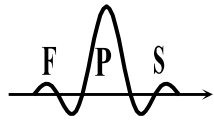


Discrete-time processing of continuous-time signals

- In our previous analysis we have admitted $y[n]=x[n]$, i.e., absence of discrete-time processing so as to show the possibility of sampling and reconstruction an analog signal. It is important to assess now the impact on the analog signal of a discrete-time processing as this is the most common scenario:



- Although it is possible/desirable to design systems where the A/D sampling frequency is different from the D/A sampling frequency, (e.g. that is the case of oversampling that is used in CD/MP3 players), we admit in this analysis that both are equal.



Discrete-time processing of continuous-time signals

- If the discrete-time system is LTI and is characterized in the frequency by $H(e^{j\omega})$, then:
$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$$

but since: $X(e^{j\omega}) = X_a(\Omega)|_{\Omega=\omega/T}$ which means:
$$X(e^{j\Omega T}) = X_a(\Omega) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_c\left(\Omega - k \frac{2\pi}{T}\right)$$

We have also seen that considering for example zero-order reconstruction, then:

$$Y_c(\Omega) = Y(e^{j\Omega T}) T \frac{\sin \Omega T / 2}{\Omega T / 2} H_r(\Omega)$$

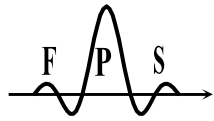
and we obtain finally:

$$Y_c(\Omega) = H(e^{j\Omega T}) \cdot \frac{\sin \Omega T / 2}{\Omega T / 2} \cdot H_r(\Omega) \sum_{k=-\infty}^{+\infty} X_c\left(\Omega - k \frac{2\pi}{T}\right)$$

we may thus conclude that:

- if the ‘anti-aliasing’ filter at the input of the system enforces $X_c(\Omega)=0$ for $|\Omega| > \pi/T$ (or if $x_c(t)$ possesses this property already), then there is no overlap of spectral images in the summation
- if the reconstruction filter eliminates spectral images for $|\Omega| > \pi/T$ and ensures $\sin(x)/x$ compensation, then the previous expression simplifies to:

$$Y_c(\Omega) = H(e^{j\Omega T}) X_c(\Omega) = H_{eff}(\Omega) X_c(\Omega)$$



Discrete-time processing of continuous-time signals

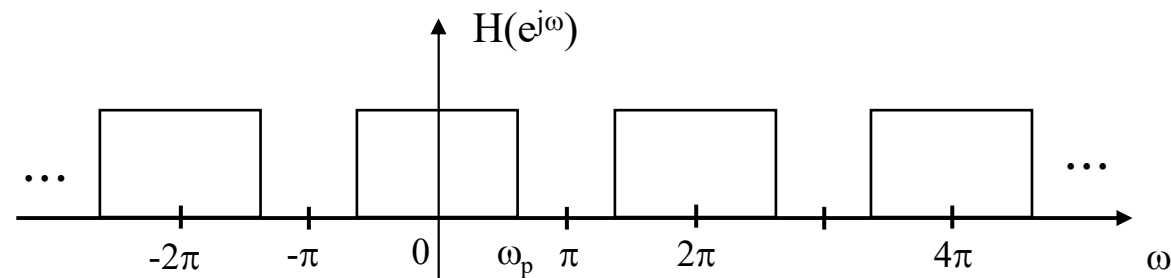
- we may finally conclude that if the discrete-time system is linear and time-invariant, from the input to the output of the system all happens as if there is an analog processing characterized by $H_{\text{eff}}(\Omega)$, whose relation to discrete-time processing is:

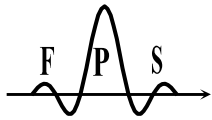
$$H_{\text{eff}}(\Omega) = \begin{cases} H(e^{j\Omega T}), & |\Omega| < \pi / T \\ 0, & |\Omega| \geq \pi / T \end{cases}$$

Example: continuous-time low-pass filtering by means of a discrete-time filter

given the filter: $H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_p \\ 0, & \omega_p < |\omega| \leq \pi \end{cases}$ whose frequency response is

ω -periodic, with period 2π :

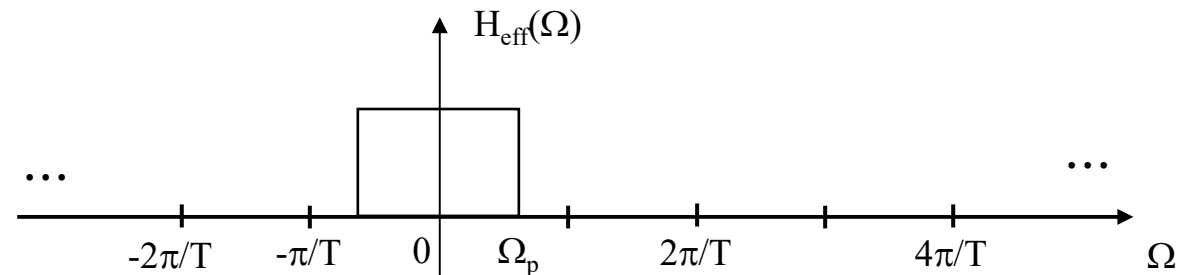




Discrete-time processing of continuous-time signals

then:

$$H_{eff}(\Omega) = \begin{cases} 1, & |\Omega T| < \omega_p \\ 0, & |\Omega T| > \omega_p \end{cases} = \begin{cases} 1, & |\Omega| < \Omega_p = \omega_p / T \\ 0, & |\Omega| > \Omega_p = \omega_p / T \end{cases}$$



A few reasons justifying that this analog filter implemented in the discrete-time domain may be preferable:

- as the cut-off frequency $\Omega_p = \omega_p / T$ depends on T , using the same system, we may vary the effective analog cut-off frequency (i.e., we have adjustable filters), by acting solely on the sampling frequency ($1/T$),
- when we need a filter with demanding specifications, involving for example very narrow transition bands, or high stop-band attenuation, or many bands with different gains and attenuations; its realization in the analog domain is difficult, probably very expensive, and highly dependent on the characteristics of the analog components, and in any case it will show a strongly non-linear phase response. Moving that filtering effort to the discrete-time domain eliminates almost completely these inconveniences. A specific case where that is true involves A/D and D/A operations, that require, respectively, “anti-aliasing” and “anti-imaging” filters, both low-pass. The analog filter specifications are ‘alleviated’ (and in certain cases no analog filtering at all is needed) transferring most of the filtering effort to the discrete/digital domain although requiring a significant increase of the sampling frequency. In the first case, (i.e. after A/D conversion), decimating digital filters are used and in the second case (i.e. before D/A conversion), interpolating digital filters are used. We will return to these topics later on !